Fractional Distance: The Topology of the Real Number Line with Applications to the Riemann Hypothesis

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Abstract

Recent analysis has uncovered a broad swath of previously unconsidered real numbers called real numbers in the neighborhood of infinity. Here we extend the catalog of the rudimentary analytical properties of all real numbers by defining a set of fractional distance functions on the real number line and studying their behavior. The main results of this paper include (1) to prove that some real numbers are greater than any natural number, (2) to develop a technique for taking a limit at infinity via the ordinary Cauchy definition reliant on the classical epsilon-delta formalism, and (3) to demonstrate an infinite number of non-trivial zeros of the Riemann zeta function in the neighborhood of infinity. The methods used in this analysis include nothing other than basic arithmetic, a little trigonometry, and Euclidean geometry. In addition to the zeros used to disprove the Riemann hypothesis in earlier work, here we present yet more zeros which independently constitute the negation of the Riemann hypothesis.

Contents

§1 Mathematical Preliminary

§1.1 Introduction

In previous work [1], we have demonstrated the existence of a broad class of previously ignored real numbers: those in the neighborhood of infinity. In the present paper, we will again demonstrate the existence of real numbers in the neighborhood of infinity. Then we will apply the tools of real mathematical analysis to show their properties with emphasis on their topological properties. In the first section of this paper, we repeat much of what appeared in Reference [1] by giving a definition for real numbers. In the second section, we define a set of functions called fractional distance functions. These functions will facilitate much of the analysis that appears in the following sections. The fourth section is dedicated most specifically to the topological properties of the real number line. In the fifth section, we disprove the Riemann hypothesis. In the sixth and final section, we make some non-rigorous and broad connections to the Modified Cosmological Model (MCM) [2]. While the main results of this paper stand on their own, the sixth section assumes at least some familiarity with the MCM. The MCM is the overarching name given to the physics-based research program whose previously defined requirements [2] constitute the underlying philosophical predicate for the present analysis of fractional distance along the real number line.

§1.2 Real Numbers

In this section, the reader is invited to recall the distinction between the real numbers $\mathbb{R}$ and the real number field $\mathcal{R} = (\mathbb{R}, +, \times)$. Numbers exist independently of their operations and here we define real numbers as cuts in the real number line. By defining a line, giving it a label "real," defining cuts in a line, and then defining real numbers as cuts in the real number line, we make a rigorous definition of real numbers sufficient for applications at any level of rigor with no circular reasoning at any step in the development of the definition.

Definition 1.2.1 A line is a 1D space extending infinitely far in both directions.

Definition 1.2.2 A number line is a line equipped with the Euclidean metric

$$L(x, y) = |y - x| \quad .$$

Definition 1.2.3 The real number line is a unique number line given the name "real."

Definition 1.2.4 The interval representation of a 1D space extending infinitely far in both directions is $(-\infty, \infty)$. In other words, $(-\infty, \infty)$ is an infinite line.

Definition 1.2.5 If x is a cut in a line then

$$(-\infty, \infty) = (-\infty, x) \cup [x, \infty) \quad .$$

Definition 1.2.6 A real number $x \in \mathbb{R}$ is a cut in the real number line.

Definition 1.2.7 Real numbers are constructed in algebraic interval notation as

$$\mathbb{R} \equiv (-\infty, \infty) \quad .$$

Definition 1.2.8 $\mathbb{R}_0$ is a subset of all real numbers

$$\mathbb{R}_0 = \{x \in \mathbb{R} \mid (\exists n \in \mathbb{N})[-n < x < n]\} \quad .$$

Here we define $\mathbb{R}_0$ as the set of all $x \in \mathbb{R}$ such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$.

Definition 1.2.9 $\mathbb{R}_\infty$ is a subset of all real numbers with the property

$$\mathbb{R}_\infty \subset \mathbb{R} \setminus \mathbb{R}_0 \quad .$$

§1.3 Affinely Extended Real Numbers

To prove in Section 2.2 that $\mathbb{R}_\infty$ is not the empty set, namely that there are real numbers larger than every natural number, we will make reference to "line segments" beyond the simpler construction called "a line." Most generally, a line with two endpoints A and B is a called a line segment $AB \equiv [a, b]$ where $[a, b]$ is an interval of numbers. Nowhere is it required that the endpoints must be real numbers so the interval $[a, b] = [-\infty, \infty]$ conforms to the definition of a line segment AB. The real line $\mathbb{R}$ together with two endpoints $\{\pm\infty\}$ is called the affinely extended real number line $\overline{\mathbb{R}} \equiv [-\infty, \infty]$. This section lays the foundation for a subsequent analysis of general line segments by giving the properties of $\overline{\mathbb{R}}$. To create $\overline{\mathbb{R}}$, we do not need to invent infinity, or conjure infinity, or introduce anything at all which was not already contained in the definition of $\mathbb{R}$. The endpoints $\{\pm\infty\}$ already exist: their existence is a concession required for the representation of $\mathbb{R}$ as the interval $(-\infty, \infty)$. These extant points $\{\pm\infty\}$ are explicitly acknowledged to exist by the round bracket notation $\mathbb{R} \equiv (-\infty, \infty)$. This notation posits that they are things which exist but are not in $\mathbb{R}$. The set of all affinely extended real numbers includes these two numbers by replacing the round brackets with square brackets. In this section, we present the main properties which distinguish $\overline{\mathbb{R}}$ from $\mathbb{R}$.

Definition 1.3.1 For $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$, we have the properties

$$\lim_{x\to 0^\pm} \frac{1}{x} = \text{diverges} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \text{diverges} \quad .$$

Definition 1.3.2 Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$, we have the properties

$$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \infty \quad .$$

Definition 1.3.3 Infinity is such that

$$\infty - \infty = \text{undefined} \quad , \qquad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Definition 1.3.4 The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 1.3.5 The affinely extended real numbers are defined in interval notation as

$$\overline{\mathbb{R}} \equiv [-\infty, \infty] \quad .$$

Definition 1.3.6 An affinely extended real number $x \in \overline{\mathbb{R}}$ is $\pm\infty$ or it is a cut in the affinely extended real number line:

$$[-\infty, \infty] = [-\infty, x) \cup [x, \infty] \quad .$$

Main Theorem 1.3.7 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 1.3.4.

§1.4 Line Segments

In this section, we review what is commonly understood regarding line segments. We begin to develop the relationship between points in a line segment and cuts in a line. During the analyses which follow in the remainder of this paper, we closely examine the differences between cuts and points. If x is a cut in a line, then

$$(a, b) = (a, x) \cup [x, b) \quad .$$

If x is a point in a line segment, then we have a tentative, preliminary understanding that

$$[a, b] = [a, x) \cup \{x\} \cup (x, b] \quad .$$

Definition 1.4.1 A line segment AB is a line together with two different endpoints $A \neq B$.

Definition 1.4.2 AB is a real line segment if the endpoints A and B bound some subset of the real line $\mathbb{R} \equiv (-\infty, \infty)$.

Definition 1.4.3 A real line segment AB is represented in interval notation as $AB \equiv [a, b]$ where a and b are any two affinely extended real numbers $a, b \in \overline{\mathbb{R}}$ such that $a < b$.

Definition 1.4.4 The notation AB is called the geometric representation of a line segment.

Definition 1.4.5 The notation $[a, b]$ is called the algebraic representation of a line segment.

Definition 1.4.6 Two line segments AB and CD are equal if and only if

$$\frac{AB}{CD} = \frac{CD}{AB} = 1 \quad .$$

Definition 1.4.7 **AB** is a special label given to the unique line segment $AB \equiv [0, \infty]$:

$$AB = \mathbf{AB} \qquad \Longleftrightarrow \qquad AB \equiv [0, \infty] \quad .$$

Definition 1.4.8 X is an interior point of AB if and only if

$$X \in AB \quad , \quad X \neq A \quad , \qquad \text{and} \qquad X \neq B \quad .$$

Definition 1.4.9 If X is an interior point of AB then

$$AB = AX + XB \quad .$$

Definition 1.4.10 Every geometric point X along a real line segment AB has one and only one algebraic interval representation $\mathscr{X}$. If $\mathscr{X}$ is the algebraic representation of X, then $X \equiv \mathscr{X}$.

Theorem 1.4.11 *If X is an interior point of a real segment AB, then X has an algebraic interval representation as one or more real numbers.*

Proof. X is an interior point of AB so

$$AB = AX + XB \quad .$$

Since $AB \equiv [a,b]$ and $(a,b) \subset \mathbb{R}$, it follows that the algebraic representation $\mathscr{X}$ of an interior point X is such that

$$x \in \mathscr{X} \qquad \Longrightarrow \qquad a < x < b \quad .$$

For $(a,b) \subset \mathbb{R}$, this inequality is only satisfied by $x \in \mathbb{R}$. ✐

Remark 1.4.12 It will be a main result of this paper to show that the infinite length of a line segment such as $\mathbf{AB} \equiv [0,\infty]$ will allow us to put more than one number into the algebraic representation $\mathscr{X}$ of a geometric point X. If a line segment has finite length $L \in \mathbb{R}_0$, we will show that there is at most one real number in the algebraic representation of one its points. However, this constraint will vanish when the line segment has infinite length.

Definition 1.4.13 The algebraic representation $\mathscr{X}$ of a geometric point X lying along a real line segment AB is

$$\mathscr{X} = [x_1, x_2] \quad , \qquad \text{where} \qquad x_1, x_2 \in \overline{\mathbb{R}} \quad .$$

The special (intuitive) case of $x_1 = x_2 = x$ gives

$$\mathscr{X} = [x,x] = \{x\} \quad .$$

Here we have expressed $\mathscr{X}$ with included endpoints x_1 and x_2. Most generally, however, an algebraic representation of a point is a single number or it is some interval of numbers, *i.e.*: all variations of (x_1, x_2), $(x_1, x_2]$, and $[x_1, x_2)$ are allowable algebraic representations of X and we do not require that $x_1 \neq x_2$ in all cases.

Definition 1.4.14 If $X \equiv \mathscr{X} = [x_1, x_2]$ with $x_1 \neq x_2$, then $x \in [x_1, x_2]$ is said to be a *possible* representation of X. If $x_1 = x_2 = x$, then x is said to be *the* algebraic representation of X. If x is the algebraic representation of X then $x \equiv X$. If x is a possible representation of X then $x \in X$.

Definition 1.4.15 If x is a possible algebraic representation of X then

$$x \in \mathscr{X} = [x_1, x_2] \equiv X \quad .$$

This statement may be abbreviated as $x \in X$ while $x \equiv X$ specifies the case of $x_1 = x_2$.

Remark 1.4.16 A point in a line segment is a set of numbers, possibly only one number, and it remains to identify the exact relationship between numbers (cuts) and geometric points. The key feature of Definition 1.4.13 is that it allows, provisionally, a many-to-one relationship between cuts in lines and points in line segments. In Section 2.4, we will strictly prove that which has been suggested: the algebraic representation of $X \in AB$ is only constrained to be a unique real number when AB has finite length.

Definition 1.4.17 A point C is a midpoint of a line segment AB if and only if

$$\frac{AC}{AB} = \frac{CB}{AB} = 0.5 \quad .$$

Alternatively, C is a midpoint of AB if and only if

$$AC = CB \quad , \qquad \text{and} \qquad AC + CB = AB \quad .$$

Definition 1.4.18 Hilbert's discarded axiom [3] states the following: Any four points $\{A, B, C, D\}$ of a line can always be labeled so that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.

Remark 1.4.19 Hilbert's discarded axiom is discarded not because it wrong, but because it is implicit in Hilbert's other axioms. It is discarded by redundancy rather than invalidity.

Theorem 1.4.20 *All line segments have at least one midpoint.*

Proof. Let there be a line segment AB and two circles of equal radii centered on the points A and B. Let the two radii be less than AB but large enough such that the circles intersect at exactly two points S and T. (This geometric configuration, Figure 1, is guaranteed to exist by Hilbert's discarded axiom pertaining to $\{A, X_1, X_2, B\}$.) It follows by construction that

$$AS = AT = BS = BT \quad .$$

Let the line segment ST intersect AB at C. By the Pythagorean theorem, C is a midpoint of AB because

$$AC^2 + CS^2 = AS^2 \quad , \qquad \text{and} \qquad BC^2 + CS^2 = BS^2 \quad ,$$

together yield

$$AC = BC \quad .$$

C separates AB into two line segments so

$$AC + CB = AB \quad .$$

These two conditions, $AC = BC$ and $AC + CB = AB$, jointly conform to Definition 1.4.17 so C is a midpoint of an arbitrary line segment AB.

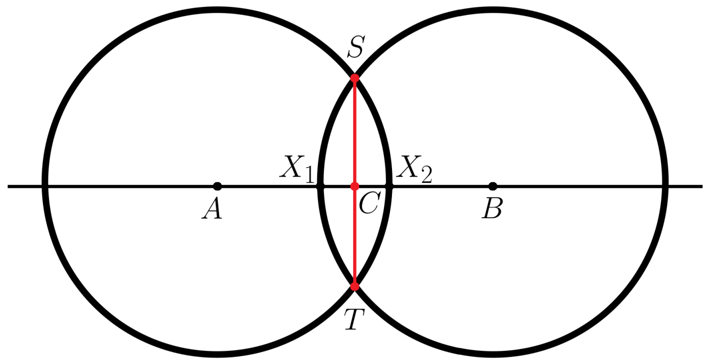

Figure 1: This figure proves that every line segment AB has one and only one midpoint.

Example 1.4.21 Theorem 1.4.20 regards an arbitrary line segment AB. Therefore, the theorem holds in the case of an arbitrary line segment AB. However, one might be afflicted with the assumption that it is impossible to define two intersecting circles centered on the endpoints of an arbitrary line segment such as $\mathbf{AB} \equiv [0, \infty]$. To demonstrate how the arbitrary case of any line segment AB generalizes to the specific case of $\mathbf{AB}$, let $AB \equiv [0, \pi/2]$ and let $x' \in \mathscr{X}$ be a number in the algebraic representation of $X \in AB$. We say that $[0, \pi/2]$ is the algebraic representation of AB charted in x'. Let x be such that

$$x = \tan(x') \quad ,$$

so that x and x' are related by a conformal transformation. Using

$$\tan(0) = 0 \quad , \qquad \text{and} \qquad \tan\left(\frac{\pi}{2}\right) = \infty \quad ,$$

where the latter follows from Definition 1.3.2, it follows that $[0, \infty]$ is the algebraic representation of AB charted in x. Therefore, $AB = \mathbf{AB}$ with respect to the x chart. Hilbert's discarded axiom guarantees the existence of two points $X_1 \in AB$ and $X_2 \in AB$ with algebraic representations $\mathscr{X}_1$ and $\mathscr{X}_2$ such that

$$x' = \frac{\pi}{6} \in \mathscr{X}_1 \quad , \qquad \text{and} \qquad x' = \frac{\pi}{3} \in \mathscr{X}_2 \quad .$$

If the radius of the circle centered on A is AX_2 and the radius of the circle centered on B is BX_1, then it is guaranteed that these circles will intersect at two points, as in Figure 1. Since $AB = \mathbf{AB}$ in the x chart, it is required that $X_1 \in \mathbf{AB}$ and $X_2 \in \mathbf{AB}$. Therefore, circles centered on the endpoints of $\mathbf{AB}$ with radii AX_2 and BX_1 will intersect at exactly two points. The chart on the line segment cannot affect the line segment's basic geometric properties. X_1 and X_2 do not cease to exist simply because we define a conformal chart $x = \tan(x')$. If they ceased to exist, then that would violate Hilbert's discarded axiom. This example demonstrates that Theorem 1.4.20 is valid for the specific case of $AB = \mathbf{AB}$.

Theorem 1.4.22 *All line segments have one and only one midpoint.*

Proof. For proof by contradiction, suppose C and D are two different midpoints of a line segment AB. C and D are midpoints of AB so we may derive from Definition 1.4.17

$$AC = CB = \frac{AB}{2} \quad , \qquad \text{and} \qquad AD = DB = \frac{AB}{2} \quad .$$

It follows that $AC = AD$. Therefore, $C = D$ and we invoke a contradiction having assumed that C and D are different.

§2 Fractional Distance

§2.1 Fractional Distance Functions

If there are two circles with equal radii whose centers are separated by an infinite distance, then what numerical radii less than infinity will allow the circles to intersect at exactly two points? To answer this question, we will introduce fractional distance functions.

Definition 2.1.1 For any point X on a real line segment AB, the geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is a continuous bijective map

$$\mathcal{D}_{AB}(AX) : AB \to \mathbb{R} \quad .$$

which takes $AX \subset AB$ and returns real numbers. This function returns AX as a fraction of AB. Emphasizing the geometric construction, the geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is defined as

$$\mathcal{D}_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{AX}{AB} & \text{for} \quad X \neq A \quad \text{and} \quad X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Remark 2.1.2 The domain of $\mathcal{D}_{AB}(AX)$ is defined as subsets of real line segments. This allows $AX = AA$ which would be excluded from a domain of real line segments because AA does not have two different endpoints.

Theorem 2.1.3 *For any point $X \in AB$, the geometric fractional distance function $\mathcal{D}_{AB}(AX) : AB \to R$ has range $R = [0, 1]$.*

Proof. To initiate proof by contradiction, assume $\mathcal{D}_{AB}(AX) < 0$. Then one of the lengths in the fraction must be negative and we invoke a contradiction with the length of a line segment defined as a positive real number. If $\mathcal{D}_{AB}(AX) > 1$, then $AX > AB$ and we invoke a contradiction because AX cannot be in the domain AB if it is larger than AB. We have excluded from R all numbers less than 0 and greater than one. Since $\mathcal{D}_{AB}(AX)$ is a continuous function taking the values 0 and 1 at the endpoints of its domain, the intermediate value theorem requires that the range of $\mathcal{D}_{AB}(AX) : AB \to R$ is $R = [0, 1]$. ∎

Theorem 2.1.4 *All line segments have at least one midpoint.*

Proof. (Reproof of Theorem 1.4.20.) $\mathcal{D}_{AB}(AX)$ is a continuous function on the domain AB taking finite values 0 and 1 at the endpoints of its domain. By the intermediate value theorem, there exists a point C in the domain AB for which $\mathcal{D}_{AB}(AC) = 0.5$. By Definition 1.4.17, C is a midpoint of AB. ∎

Theorem 2.1.5 *Every midpoint of a line segment AB is an interior point of AB.*

Proof. If X is not an interior point of AB, then $X = A$ or $X = B$. In each case respectively, the geometric fractional distance function returns

$$\mathcal{D}_{AB}(AA) = 0 \quad , \qquad \text{or} \qquad \mathcal{D}_{AB}(AB) = 1 \quad .$$

A point C is a midpoint of AB if and only if

$$\mathcal{D}_{AB}(AC) = 0.5 \quad .$$

No midpoint can be an endpoint. ∎

Remark 2.1.6 Given the geometric fractional distance function, it is not clear how to compute $\mathcal{D}_{AB}(AX)$ when X is an arbitrary interior point. By Definition 2.1.1, we know that the fraction AX/AB is a real number but we have not developed any tools for finding the numerical value. The quotient notation required for computing fractional distance calls for an algebraic notion of distance.

Definition 2.1.7 $\mathcal{D}^{\dagger}_{AB}(AX)$ is the algebraic fractional distance function. It is an algebraic expression which totally replicates the behavior of the geometric fractional distance function $\mathcal{D}_{AB}(AX)$ on an arbitrary line segment $AB \equiv [a, b]$.

Definition 2.1.8 The algebraic fractional distance function $\mathcal{D}^{\dagger}_{AB}(AX)$ is constrained to be such that

$$\mathcal{D}^{\dagger}_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

for every point $X \in AB$.

Definition 2.1.9 An algebraic fractional distance function of the first kind

$$\mathcal{D}'_{AB}(AX) : AB \to [0, 1] \quad ,$$

is a map on subsets of real line segments

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} & \text{for} \quad X \neq A \quad \text{and} \quad X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad ,$$

where

$$\frac{\|AX\|}{\|AB\|} \equiv \frac{\text{len}[a, x]}{\text{len}[a, b]} \quad ,$$

and $[a, x]$ and $[a, b]$ are the line segments AX and AB expressed in interval notation.

Definition 2.1.10 The norm $\|AX\| = \text{len}[a, x]$ which appears in $\mathcal{D}'_{AB}(AX)$ is defined so that

$$\mathcal{D}'_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

Definition 2.1.11 An algebraic fractional distance function of the second kind

$$\mathcal{D}''_{AB}(AX) : [a, b] \to [0, 1] \quad ,$$

is a map on intervals of the form

$$\mathcal{D}''_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\text{len}[a, x]}{\text{len}[a, b]} & \text{for} \quad X \neq A \quad \text{and} \quad X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Remark 2.1.12 The purpose in defining two kinds of algebraic fractional distance functions (FDFs) is so that we may compare their properties and then choose the one that exactly replicates the behavior of the geometric FDF $\mathcal{D}_{AB}(AX)$. By Definition 2.1.8, this is required for the algebraic FDF $\mathcal{D}^{\dagger}_{AB}(AX)$.

Definition 2.1.13 The ordering of $\mathbb{R}$ is such that for any $x, y \in \mathbb{R}$, if

$$x \in [x_1, x_2] = \mathscr{X} \equiv X \quad , \qquad \text{and} \qquad y \in [y_1, y_2] = \mathscr{Y} \equiv Y \quad ,$$

then

$$\mathcal{D}_{AB}(AX) > \mathcal{D}_{AB}(AY) \quad \implies \quad x > y \quad .$$

Theorem 2.1.14 *The geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is injective (one-to-one) on all real line segments.*

Proof. By Definition 2.1.1, the geometric FDF is

$$\mathcal{D}_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{AX}{AB} & \text{for} \quad X \neq A \quad \text{and} \quad X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

For proof by contradiction, assume $\mathcal{D}_{AB}(AX)$ is not always injective. Then there exists some $X' \neq X''$ such that

$$\frac{AX'}{AB} = \frac{AX''}{AB} \quad .$$

The range of $\mathcal{D}_{AB}(AX)$ is $[0, 1]$ and it is known that all such $0 \leq x \leq 1$ have an additive inverse element. This allows us to write

$$0 = \frac{AX''}{AB} - \frac{AX'}{AB} = \frac{AX'' - AX'}{AB} \quad \iff \quad AX'' = AX' \quad .$$

This contradicts the assumed condition $X' \neq X''$. The geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is injective on all real line segments. ∎

Remark 2.1.15 In Theorem 2.1.14, we have not considered specifically the case in which AB is a line segment of infinite length. There are many numbers x' and x'' such that 0 being equal to their difference divided by infinity does not imply that $x' = x''$:

$$0 = \frac{x' - x''}{\infty} \quad \not\Leftrightarrow \quad x' = x'' \quad . \tag{2.1}$$

However, $\mathcal{D}_{AB}(AX)$ does not have numbers in its domain. The fraction in Equation (2.1) can never appear when computing AX/AB because $\mathcal{D}_{AB}(AX)$ takes line segments or simply the point A (written as AA in abused line segment notation.) To be clear, simplifying the expression $\mathcal{D}_{AB}(AX)$ in the general case requires some supplemental constraint like $AB = cAX$ for some scalar

c. Numbers such as the ∞ in the denominator of Equation (2.1) will be used only to compute $\mathcal{D}^{\dagger}_{AB}(AX)$ when we introduce the norm $\|AX\|$. The reliance on numerical representations of line segments is the main feature distinguishing the algebraic FDF from the geometric FDF.

Theorem 2.1.16 *The geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is surjective (onto) on all real line segments.*

Proof. Given the range $R = [0, 1]$, proof follows from the notion of geometric fractional distance.

Remark 2.1.17 Now that we have shown the elementary properties of the geometric FDF, we will examine the similar behaviors of the algebraic FDFs of the first and second kinds.

Conjecture 2.1.18 The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}(AX)$ is injective (one-to-one) on all real line segments. (This is proven in Theorem 2.3.22.)

Theorem 2.1.19 *The algebraic fractional distance function of the second kind $\mathcal{D}''_{AB}(AX)$ is not injective (one-to-one) on all real line segments.*

Proof. Recall that Definition 2.1.11 gives $\mathcal{D}''_{AB}(AX) : [a, b] \to [0, 1]$ as

$$\mathcal{D}''_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\text{len}[a,x]}{\text{len}[a,b]} & \text{for} \quad X \neq A \quad \text{and} \quad X \neq B \\ 0 & \text{for} \quad X = A \end{cases} .$$

Injectivity requires that

$$\mathcal{D}''_{AB}(AX_1) = \mathcal{D}''_{AB}(AX_2) \quad \iff \quad [a, x_1] = [a, x_2] \quad \iff \quad x_1 = x_2 \ .$$

Let $n_1, n_2 \in \mathbb{N}$ and $n_1 \neq n_2$ such that $n_1 \in \mathcal{N}_1 \equiv N_1$ and $n_2 \in \mathcal{N}_2 \equiv N_2$. We have

$$\mathcal{D}''_{\mathbf{AB}}(AN_1) = \frac{\text{len}[0, n_1]}{\text{len}[0, \infty]} = 0 \ , \qquad \text{and} \qquad \mathcal{D}''_{\mathbf{AB}}(AN_2) = \frac{\text{len}[0, n_1]}{\text{len}[0, \infty]} = 0 \ .$$

Therefore, the algebraic fractional distance function of the second kind is not injective on all real line segments.

Remark 2.1.20 At this point, we can rule out $\mathcal{D}''_{AB}(AX)$ as the definition of $\mathcal{D}^{\dagger}_{AB}(AX)$. This follows because the geometric FDF $\mathcal{D}_{AB}(AX)$ which constrains $\mathcal{D}^{\dagger}_{AB}(AX)$ is one-to-one. If $\mathcal{D}_{AB}(AX)$ is one-to-one on all real line segments AB then so is $\mathcal{D}^{\dagger}_{AB}(AX)$. Carefully note that the domain of the algebraic FDF of the first kind is line segments rather than algebraic intervals. We have

$$\mathcal{D}'_{AB}(AX) : AB \to [0,1] \quad , \qquad \text{and} \qquad \mathcal{D}''_{AB}(AX) : [a,b] \to [0,1] \quad .$$

Taking for granted that we will prove the injectivity of $\mathcal{D}'_{AB}(AX)$ in Theorem 2.3.22, this distinction of domain will prohibit the breakdown in the one-to-one property when a point $X \in AB$ can have many different numbers in its algebraic representation. Assuming that the domain of the algebraic FDF is an algebraic interval $[a,b]$ is likely the root cause of **much pathology in modern analysis**.

Theorem 2.1.21 *The geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is continuous everywhere on the domain* $\mathbf{AB}$.

Proof. To prove that $\mathcal{D}_{AB}(AX)$ is continuous on $\mathbf{AB}$, it will suffice to show that $\mathcal{D}_{\mathbf{AB}}(AX)$ is continuous at the endpoints and an interior point.

• (*Interior point*) A function $f(x)$ is continuous at an interior point x_0 of its domain $[a,b]$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad .$$

In terms of the geometric FDF, the statement that $\mathcal{D}_{\mathbf{AB}}(AX)$ is continuous at an interior point $X' \in \mathbf{AB}$ becomes

$$\lim_{X \to X'} \mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{D}_{\mathbf{AB}}(AX') \quad .$$

Obviously, $\mathcal{D}_{\mathbf{AB}}(AX)$ satisfies the definition of continuity on the interior of $\mathbf{AB}$.

• (*Endpoint A*) A function $f(x)$ is continuous at the endpoint a of its domain $[a,b]$ if and only if

$$\lim_{x \to a^+} f(x) = f(a) \quad .$$

We conform to this definition of continuity with

$$\lim_{X \to A^+} \mathcal{D}_{\mathbf{AB}}(AX) = \lim_{X \to A^+} \frac{AX}{\mathbf{AB}} = \frac{AA}{\mathbf{AB}} = \mathcal{D}_{\mathbf{AB}}(AA) \quad .$$

• (*Endpoint B*) A function $f(x)$ is continuous at the endpoint b of its domain $[a, b]$ if and only if

$$\lim_{x \to b^-} f(x) = f(b) \quad .$$

We conform to this definition with

$$\lim_{X \to B^-} \mathcal{D}_{\mathbf{AB}}(AX) = \lim_{X \to B^-} \frac{AX}{\mathbf{AB}} = \frac{AB}{\mathbf{AB}} = \mathcal{D}_{\mathbf{AB}}(AB) \quad .$$

The geometric FDF is continuous everywhere on its domain.

Theorem 2.1.22 *The algebraic fractional distance function of the first kind* $\mathcal{D}'_{AB}(AX)$ *is not continuous everywhere on the domain* $\mathbf{AB}$.

Proof. A function $f(x)$ with domain $x \in [a, b]$ is continuous at b if

$$\lim_{x \to b^-} f(x) = f(b) \quad ,$$

In terms of $\mathcal{D}'_{AB}(AX)$, the statement that $\mathcal{D}'_{\mathbf{AB}}(AX)$ is continuous at B becomes

$$\lim_{X \to B} \mathcal{D}'_{\mathbf{AB}}(AX) = \mathcal{D}'_{\mathbf{AB}}(AB) = 1 \quad .$$

Evaluation yields

$$\lim_{X \to B} \mathcal{D}'_{\mathbf{AB}}(AX) = \lim_{x \to \infty} \frac{\text{len}[0, x]}{\text{len}[0, \infty]} = \lim_{x \to \infty} \frac{1}{\infty} x = \lim_{x \to \infty} 0 \neq 1 = \mathcal{D}'_{\mathbf{AB}}(AB) \quad .$$

The algebraic FDF of the first kind is not continuous everywhere on all real line segments.

Remark 2.1.23 In Theorem 2.1.22, we have shown that the limit approaches 0 rather than the value 1 required for $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AB)$ to agree with $\mathcal{D}_{\mathbf{AB}}(AB)$. However, we may also write this limit as

$$\lim_{x \to \infty} \frac{1}{\infty} x = \lim_{\substack{x \to \infty \\ y \to \infty}} \frac{x}{y} = \lim_{y \to \infty} \infty \frac{1}{y} = \lim_{y \to \infty} \infty = \infty \quad .$$

By rewriting the limit, we have shown that it approaches ∞ rather than 0. Perhaps, then, it would be better to write simply

$$\lim_{x \to \infty} \frac{x}{\infty} = \frac{\infty}{\infty} = \text{undefined} \neq 1 \quad .$$

In any case, we have shown that no elementary evaluation of the limit produces the correct limit at infinity. Therefore, we should also examine the Cauchy definition of the limit relying on the ε–δ formalism.

Theorem 2.1.24 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}(AX)$ does not converge to a Cauchy limit at infinity.*

Proof. According to the Cauchy definition of the limit of $f(x) : D \to R$ at ∞, we say that

$$\lim_{x\to\infty} f(x) = L \quad ,$$

if and only if

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \qquad \text{s.t} \qquad \forall x \in D \quad ,$$

we have

$$0 < |x - \infty| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon \quad .$$

There is no $\delta > \infty$ so $\mathcal{D}'_{AB}(AX)$ fails the Cauchy criterion for convergence to a limit at infinity.

Remark 2.1.25 In general, the above Cauchy definition of a limit fails for any limit at infinity because there is never a δ greater than infinity. It is a main result of this paper that we will develop a technique for taking limits at infinity with the normal Cauchy prescription. This result appears in Section 3.1.

Paradox 2.1.26 The algebraic FDF $\mathcal{D}^{\dagger}_{AB}(AX)$ exists by definition. It is a function which has every behavior of the geometric FDF $\mathcal{D}_{AB}(AX)$ and also adds the ability to compute numerical ratios between the lengths of two line segments. Numbers being generally within the domain of algebra, the geometric FDF returns a fraction that we have no way to simplify. Since it is hard to conceive of an irreducible analytic form for the algebraic FDF other than $\mathcal{D}'_{AB}(AX)$ and $\mathcal{D}''_{AB}(AX)$, it is paradoxical that neither of them replicate the global behavior of the algebraic FDF $\mathcal{D}^{\dagger}_{AB}(AX)$. After developing some more material, we will show in Section 3.3 that $\mathcal{D}^{\dagger}_{AB}(AX)$ is $\mathcal{D}'_{AB}(AX)$ after all. We will prevent an unwarranted assumption about infinity from sneakily propagating into the present analysis, and then we will fix the discrepancy in the continuity of $\mathcal{D}^{\dagger}_{AB}(AX)$ and $\mathcal{D}'_{AB}(AX)$.

Theorem 2.1.27 *If x is a real number in the algebraic representations of both $X \in AB$ and $Y \in AB$, then $X = Y$.*

Proof. If $X \neq Y$ then

$$\mathcal{D}^{\dagger}_{AB}(AX) \neq \mathcal{D}^{\dagger}_{AB}(AY) \quad .$$

If $x \in X$ and $x \in Y$, then it is possible to make cuts at X and Y such that

$$\mathcal{D}^{\dagger}_{AB}(AX) = \frac{\text{len}[a,x]}{\text{len}[a,b]} = \mathcal{D}^{\dagger}_{AB}(AY) \quad .$$

This contradiction requires $X = Y$. ✐

§2.2 Comparison of $\mathbb{R}$ and $\mathbb{N}$

The main result of this section is to prove definitively that there exist real numbers greater than any natural number, *i.e.*: $\mathbb{R}_\infty$ is not the empty set.

Definition 2.2.1 Every interval has a number at its center. The number at the center of an interval $[a, b]$ is the the average of a and b. This holds for all intervals $[a, b)$, $(a, b]$, and (a, b).

Theorem 2.2.2 *There exists a unique real number halfway between* 0 *and* ∞*.*

Proof. By Theorem 1.4.22 and Definition 1.4.17, there exists one midpoint C of every line segment AB such that

$$\mathcal{D}_{AB}(AC) = 0.5 \quad .$$

Recalling that we have defined $\mathcal{D}_{AB}(AX) = \mathcal{D}^\dagger_{AB}(AX)$ for all $X \in AB$, and recalling that $\mathbf{AB} \equiv [0, \infty]$, it follows that

$$\mathcal{D}^\dagger_{\mathbf{AB}}(AC) = 0.5 \quad .$$

Using $C \equiv \mathscr{C} = [c_1, c_2]$, Definition 1.4.13 requires

$$\mathbf{AB} = AC + CB \qquad \Longleftrightarrow \qquad [0, \infty] = [0, c_1) \cup \mathscr{C} \cup (c_2, \infty] \quad .$$

It follows that

$$\mathscr{C} \subset \mathbb{R} \quad .$$

Every possible number that can be in the algebraic representation of the point C is a real number. If $c_1 = c_2 = c$, then $c \in \mathbb{R}$ is the unique real number halfway between 0 and ∞. If $c_1 \neq c_2$, then by Definition 2.2.1 the average of c_1 and c_2 is the unique real number halfway between 0 and ∞. ✐

Remark 2.2.3 How can $\mathcal{D}^\dagger_{\mathbf{AB}}(AC) = 0.5$ when

$$\mathcal{D}^\dagger_{\mathbf{AB}}(AC) = \frac{\text{len}[0, c]}{\infty} \quad ?$$

The prevailing assumption about infinity is

$$x \in \mathbb{R} \qquad \Longrightarrow \qquad \frac{x}{\infty} = 0 \quad . \tag{2.2}$$

If this is true, then either (a) there exists a line segment without a midpoint or (b) the geometric and algebraic fractional distance functions do not agree for every X in an arbitrary AB. Every line segment does have a midpoint

(Theorem 1.4.22) and our fractional distance functions are defined to always agree (Definition 2.1.8). Therefore, Equation (2.2) must be reformulated as

$$x \in \mathbb{R}^0_\aleph \quad \implies \quad \frac{x}{\infty} = 0 \quad . \tag{2.3}$$

It is easy to amend the existing framework of analysis to allow non-vanishing fractions of infinity with previously neglected real numbers $\mathbb{R}_\infty$. In Section 2.3, we will define $\mathbb{R}^{\mathcal{X}}_\aleph$ such that $\mathbb{R}^0_\aleph$ satisfies Equation (2.3). Regarding Theorem 2.2.2, for the real numbers at the midpoint of $[0,\infty]$ we will define

$$x \in \mathbb{R}^{0.5}_\aleph \quad \implies \quad \frac{x}{\infty} = 0.5 \quad .$$

In addition to motivating the $\mathbb{R}^{\mathcal{X}}_\aleph$ notation which will be introduced in Section 2.3, the present remark illustrates the reasoning behind allowing geometric points to be represented as entire intervals $X \equiv \mathscr{X}$: Many real numbers divided by ∞ give 0 but only the left endpoint of $\mathbf{AB}$ will have vanishing fractional distance.

Main Theorem 2.2.4 *Some elements of* $\mathbb{R}$ *are greater than every element of* $\mathbb{N}$.

Proof. Let $n_1, n_2 \in \mathbb{N}$ and $[n_1, n_2] = \mathscr{N} \equiv N \in \mathbf{AB}$. Then

$$\min[\mathcal{D}^\dagger_{\mathbf{AB}}(AN)] = \min \frac{\|AN\|}{\|\mathbf{AB}\|} = \frac{\text{len}[0,n_1]}{\text{len}[0,\infty]} = \frac{n_1}{\infty} = 0 \quad ,$$

and

$$\max[\mathcal{D}^\dagger_{\mathbf{AB}}(AN)] = \max \frac{\|AN\|}{\|\mathbf{AB}\|} = \frac{\text{len}[0,n_2]}{\text{len}[0,\infty]} = \frac{n_2}{\infty} = 0 \quad ,$$

(Purely for illustrative purposes, here we have added the $\|AX\|$ notation which is, as yet, only assigned properly to the algebraic fractional distance function of the first type $\mathcal{D}'_{\mathbf{AB}}(AX)$.) $\mathcal{D}^\dagger_{AB}(AX)$ is a continuous function with

$$\mathcal{D}^\dagger_{\mathbf{AB}}(AA) = 0 \quad , \qquad \text{and} \qquad \mathcal{D}^\dagger_{\mathbf{AB}}(AB) = 1 \quad ,$$

so the intermediate value theorem requires that it take every value between 0 and 1 for some $X \in \mathbf{AB}$. By Theorem 1.4.11, every such X has an algebraic representation as one or more real numbers. Therefore, there exist real numbers $x \in [x_1, x_2] = \mathscr{X} \equiv X$ such that

$$\mathcal{D}^\dagger_{\mathbf{AB}}(AX) > \mathcal{D}^\dagger_{\mathbf{AB}}(AN) = 0 \quad .$$

By the ordering of $\mathbb{R}$ (Definition 2.1.13), we find that there are real numbers $x \in \mathbb{R}$ greater than any natural number.

Alternatively, let **AB** have a midpoint C so that $\mathcal{D}_{\mathbf{AB}}(AC) = 0.5$. Then every real number $c \in [c_1, c_2] \equiv C$ is greater than any $n \in \mathbb{N}$ because $\mathcal{D}_{\mathbf{AB}}(AN) = 0$.

Corollary 2.2.5 $\mathbb{R}_\infty \subset \mathbb{R} \setminus \mathbb{R}_0$ *is not the empty set.*

Proof. Definition 1.2.8 defines $\mathbb{R}_0$ as the subset of $\mathbb{R}$ less than some element of $\mathbb{N}$. We have proven in Main Theorem 2.2.4 that some elements of $\mathbb{R}$ are not in $\mathbb{R}_0$. It follows that

$$\mathbb{R}_\infty \not\equiv \emptyset \ , \qquad \text{because} \qquad \mathbb{R}_\infty \equiv \mathbb{R} \setminus \mathbb{R}_0 \ .$$

§2.3 Neighborhoods of $\mathbb{R}$

In this section, we will develop notation useful for describing real numbers whose fractional magnitude with respect to infinity is larger than zero.

Definition 2.3.1 $\mathbb{R}_\aleph^\mathcal{X}$ is a subset of all real numbers

$$\mathbb{R}_\aleph^\mathcal{X} = \{x \in \mathbb{R} \mid x \geq 0 \ , \ x \in X \ , \ \mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X}\} \ .$$

Remark 2.3.2 We will treat the special case of $\mathbb{R}_\aleph^\mathcal{X}$ with $\mathcal{X} = 1$ in Section 3.2.

Definition 2.3.3 A number x is said to be greater than infinity if $x \in \mathbb{R}_\aleph^\mathcal{X}$ and $\mathcal{X} > 1$. Such numbers are not real numbers.

Definition 2.3.4 If $x \in \mathbb{R}_\aleph^\mathcal{X}$, then x is said to be in the neighborhood of numbers that are $(100 \times \mathcal{X})\%$ of the way down the real number line. (This convention ignores the negative branch of $\mathbb{R}$.)

Definition 2.3.5 A real number x is said to be in the neighborhood of the origin

$$x \in X \ , \qquad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AX) = 0 \ .$$

Definition 2.3.6 A real number x is said to be in the neighborhood of infinity

$$x \in X \ , \qquad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AX) > 0 \ .$$

Remark 2.3.7 Definition 1.2.9 states that $\mathbb{R}_\infty \subset \mathbb{R} \setminus \mathbb{R}_0$. We have not given any definition implying

$$\mathbb{R} \equiv \mathbb{R}_\infty \cup \mathbb{R}_0 \ .$$

The statement implied by the given definitions is

$$\mathbb{R} \equiv \mathbb{R}_\infty \cup \mathbb{R}_\aleph^0 \quad .$$

We have left room judiciously for numbers in the neighborhood of the origin which are larger than any natural number.

Definition 2.3.8 By Definition 2.2.1, every interval has a number at its center. If $\mathbb{R}_\aleph^\mathcal{X} \equiv (a,b)$, then the number at the center of (a,b) is $\aleph_\mathcal{X}$.

Definition 2.3.9 If $\mathcal{X} \geq 1$, then $\aleph_\mathcal{X} = \infty$.

Definition 2.3.10 If $\mathcal{X} = 0$, then $\aleph_\mathcal{X}$ is undefined.

Remark 2.3.11 When the symbol $\aleph_\mathcal{X}$ appears, it should be assumed that $0 < \mathcal{X} < 1$, which is to say that it should assumed that an unspecified $\aleph_\mathcal{X}$ is a finite, positive real number. The case of $\mathcal{X} = 0$ is not allowed and the case of $\mathcal{X} \geq 1$ is better written as ∞.

Definition 2.3.12 An alternative definition for $\mathbb{R}_\aleph^\mathcal{X}$ valid in the neighborhood of infinity is

$$\mathbb{R}_\aleph^\mathcal{X} = \{\aleph_\mathcal{X} \pm b \mid \mathcal{X} > 0 \ , \ b \in \mathbb{R}_\aleph^0\} \quad .$$

Remark 2.3.13 We have not proven that every number $x \in X$ for which $\mathcal{D}_{AB}(AX) = 0$ is an $\mathbb{R}_0$ number. Therefore, we should not unnecessarily restrict $b \in \mathbb{R}_0$ in Definition 2.3.12. For $x = 0$ in the case of $x \in A$, we have $\mathcal{D}_{\mathbf{AB}}(AA) = 0$, and this is also true for any $x \in \mathbb{R}_\aleph^0$ when $x \in A$. This means that the absolute value of $\mathbb{R}_\aleph^0$ numbers is so small that adding them to other numbers $x \in \mathbb{R}$ does not change the fractional magnitude with respect to **AB**. Specifically, if $b \in \mathbb{R}_\aleph^0$, then x and $x + b$ both have the same fractional magnitude with respect to infinity.

Definition 2.3.14 The arithmetic operations of $\aleph_\mathcal{X} \in \mathbb{R}$ with $x \in \mathbb{R}_0$ are

$$\begin{aligned}
-(\pm\aleph_\mathcal{X}) &= \mp\aleph_\mathcal{X} \\
\pm\aleph_{(-\mathcal{X})} &= \mp\aleph_\mathcal{X} \\
\pm\aleph_\mathcal{X} + x &= x \pm \aleph_\mathcal{X} \\
\aleph_\mathcal{X} \cdot x = x \cdot \aleph_\mathcal{X} &= \aleph_{(x \cdot \mathcal{X})} \\
\frac{\aleph_\mathcal{X}}{x} &= \begin{cases} \text{undefined} & \text{for} \quad x = 0 \\ \aleph_{(\mathcal{X}/x)} & \text{for} \quad x \neq 0 \end{cases} \\
\frac{x}{\aleph_\mathcal{X}} &= 0 \quad .
\end{aligned}$$

Remark 2.3.15 It possible show contradictions involving $\aleph_{(x\cdot\mathcal{X})}$ and $\aleph_{(\mathcal{X}/x)}$ when $\mathcal{X} \geq 1$. The case of $\mathcal{X} \geq 1$ is ruled out on the left hand side of the equations in Definition 2.3.14 by $\aleph_{\mathcal{X}} \in \mathbb{R}$ but $\mathcal{X} \geq 1$ is allowed on the right side. The reader's patience is begged as we will resolve these issues forthwith in Section 3.1.

Theorem 2.3.16 *The quotient of $x \in \mathbb{R}_0$ divided by any $\aleph_{\mathcal{X}} \in \mathbb{R}$ with $\mathcal{X} > 0$ is identically zero.*

Proof. We will use proof by contradiction. Let z be some positive real number such that

$$\frac{x}{\aleph_{\mathcal{X}}} = z \quad .$$

By Definition 2.1.13, we have $\|x\| < \|\aleph_{\mathcal{X}}\|$. This implies through the definition of the quotient that $\|z\| < 1$. This further implies that $z \in \mathbb{R}_0$. All such non-zero numbers have a multiplicative inverse so

$$\frac{x}{z\aleph_{\mathcal{X}}} = 1 \qquad \Longleftrightarrow \qquad x = z\aleph_{\mathcal{X}} \quad .$$

Definition 2.3.14 gives $z\aleph_{\mathcal{X}} = \aleph_{(z\mathcal{X})}$ so

$$x = \aleph_{z\mathcal{X}} = \aleph_{\mathcal{X}'} \quad , \qquad \text{where} \qquad \mathcal{X}' > 0 \quad .$$

This delivers a contradiction because it requires $x \in \mathbb{R}_{\aleph}^{\mathcal{X}'}$ for $\mathcal{X}' > 0$ while we have already defined x to be $x \in \mathbb{R}_0 \subset \mathbb{R}_{\aleph}^0$. If $z = 0$, then it has no multiplicative inverse and the contradiction is avoided. ∎

Theorem 2.3.17 $\aleph_{\mathcal{X}} \in \mathbb{R}$ *divided by infinity is equal to* $\mathcal{X}$.

Proof. By the definition of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ (Definition 2.3.12), we have $\aleph_{\mathcal{X}} \in \mathbb{R}_{\aleph}^{\mathcal{X}}$. By Definition 2.3.1, it is the property of all $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ that

$$x \in X \qquad \Longrightarrow \qquad \mathcal{D}^{\dagger}_{\mathbf{AB}}(AX) = \mathcal{X} \quad .$$

Therefore,

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \mathcal{X} \quad .$$ ∎

Theorem 2.3.18 *The product of* $\aleph_{\mathcal{X}}$ *with 0 is undefined.*

Proof. By Theorem 2.3.17, we have

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \mathcal{X} \quad .$$

This can be rewritten as

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \frac{1}{\infty}\aleph_{\mathcal{X}} = 0 \cdot \aleph_{\mathcal{X}} \quad .$$

If this operation is defined in the usual way, namely $0 \cdot \aleph_{\mathcal{X}} = 0$ then that will contradict Theorem 2.3.17 giving

$$\mathcal{X} = 0 \quad .$$

If we forced agreement with Theorem 2.3.17 through a contrived and unprecedented definition such that

$$0 \cdot \aleph_{\mathcal{X}} = \mathcal{X} \quad ,$$

then we would induce a contradiction with the multiplicative inverse of $\aleph_{\mathcal{X}}$ between

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{X}}} = 1 \quad , \qquad \text{and} \qquad \aleph_{\mathcal{X}} \cdot \frac{1}{\aleph_{\mathcal{X}}} = \aleph_{\mathcal{X}} \cdot 0 = \mathcal{X} \quad .$$

Therefore, the product of $\aleph_{\mathcal{X}}$ and 0 must be undefined.

Alternatively, if the product with zero was allowed then $0 \cdot \aleph_{\mathcal{X}} = \aleph_0$ would contradict Definition 2.3.10 which states that $\aleph_0$ is undefined.

Example 2.3.19 This example demonstrates an aspect of consistency in the arithmetic of $\aleph_{\mathcal{X}}$. By Definition 2.3.14, we have

$$\frac{\aleph_{\mathcal{X}}}{x} = \aleph_{(\mathcal{X}/x)} \quad .$$

Multiplying by $(\aleph_{\mathcal{X}})^{-1}$ on both sides yields

$$\frac{1}{x} = \frac{\aleph_{(\mathcal{X}/x)}}{\aleph_{\mathcal{X}}} \quad .$$

By Definition 2.3.23, the right hand side becomes

$$\frac{1}{x} = \frac{\aleph_{(\mathcal{X}/x)}}{\aleph_{\mathcal{X}}} = \frac{\mathcal{X}/x}{\mathcal{X}} = \frac{1}{x} \quad .$$

As a final manipulation, each of $x, \aleph_{\mathcal{X}} \in \mathbb{R}$ have a multiplicative inverse so

$$\frac{\aleph_{\mathcal{X}}}{x} = \aleph_{(\mathcal{X}/x)} \qquad \Longrightarrow \qquad \frac{x}{\aleph_{\mathcal{X}}} = \frac{1}{\aleph_{(\mathcal{X}/x)}} = 0 \quad .$$

The final equality to 0 follows from Theorem 2.3.16.

Theorem 2.3.20 *For any point $X \equiv \mathscr{X} = [x_1, x_2]$ in a real line segment AB, we have $x_1 \in \mathbb{R}_\aleph^{\mathcal{X}_0}$ if and only if $x_2 \in \mathbb{R}_\aleph^{\mathcal{X}_0}$.*

Proof. Suppose
$$x_1 \in \mathbb{R}_\aleph^{\mathcal{X}_1} \quad , \qquad \text{and} \qquad x_2 \in \mathbb{R}_\aleph^{\mathcal{X}_2} \quad .$$
By Definition 2.3.1, we find that
$$\min[\mathcal{D}^\dagger_{\mathbf{AB}}(AX)] = \frac{\text{len}[0,x_1]}{\text{len}[0,\infty]} = \frac{x_1}{\infty} = \mathcal{X}_1 \quad ,$$
and
$$\max[\mathcal{D}^\dagger_{\mathbf{AB}}(AX)] = \frac{\text{len}[0,x_2]}{\text{len}[0,\infty]} = \frac{x_2}{\infty} = \mathcal{X}_2 \quad .$$
It follows from the identity $\mathcal{D}^\dagger_{\mathbf{AB}}(AX) = \mathcal{D}_{\mathbf{AB}}(AX)$ that
$$\min \mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X}_1 \quad , \qquad \text{and} \qquad \max \mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X}_2 \quad .$$
By Definition 2.1.1, $\mathcal{D}_{\mathbf{AB}}(AX)$ is one-to-one which requires
$$\mathcal{X}_1 = \mathcal{X}_2 \quad .$$
It is obvious that the result holds for AB having finite length. ⌀

Remark 2.3.21 When defining $\mathcal{D}'_{AB}(AX)$ and $\mathcal{D}''_{AB}(AX)$ in Section 2.1, we were able to show that $\mathcal{D}''_{AB}(AX)$ is not one-to-one but we did not have to tools to prove that $\mathcal{D}'_{AB}(AX)$ is one-to-one on all real line segments. We conjectured it with Conjecture 2.1.18, and now we will prove that conjecture.

Theorem 2.3.22 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}(AX)$ is injective (one-to-one) on all real line segments.*

Proof. (Proof of Conjecture 2.1.18.) Recall that $\mathcal{D}'_{AB}(AX) : AX \to [0,1]$ is
$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} = \dfrac{\text{len}[a,x]}{\text{len}[a,b]} & \text{for} \quad X \neq A \quad \text{and} \quad X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$
Injectivity requires that
$$\mathcal{D}'_{AB}(AX_1) = \mathcal{D}'_{AB}(AX_2) \quad \iff \quad AX_1 = AX_2 \quad \iff \quad X_1 = X_2 \quad .$$
Even if there is an entire interval of numbers in the algebraic representations of each of X_1 and X_2, we have by Theorem 2.3.20:
$$\min[\mathcal{D}'_{AB}(AX_0)] = \max[\mathcal{D}'_{AB}(AX_0)] = \mathcal{X}_0 \quad .$$

This tells us that choosing any $x \in \mathscr{X} \equiv X$ will yield the same $\mathcal{D}'_{AB}(AX)$. Therefore, the injectivity of $\mathcal{D}'_{AB}(AX)$ follows from the injectivity of $\mathcal{D}_{AB}(AX)$ through the constraint

$$\mathcal{D}'_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

Definition 2.3.23 The arithmetic operations of $\aleph_{\mathcal{X}}$ numbers with other $\aleph_{\mathcal{X}}$ numbers are

$$\begin{aligned} -(\pm\aleph_{\mathcal{X}_1} + \aleph_{\mathcal{X}_2}) &= \mp\aleph_{\mathcal{X}_1} - \aleph_{\mathcal{X}_2} \\ \pm\aleph_{\mathcal{X}_1} + \aleph_{\mathcal{X}_2} &= \aleph_{(\mathcal{X}_2 \pm \mathcal{X}_1)} \\ \aleph_{\mathcal{X}_1} \cdot \aleph_{\mathcal{X}_2} = \aleph_{\mathcal{X}_2} \cdot \aleph_{\mathcal{X}_1} &= \infty \\ \frac{\aleph_{\mathcal{X}_1}}{\aleph_{\mathcal{X}_2}} = \left(\frac{\aleph_{\mathcal{X}_2}}{\aleph_{\mathcal{X}_1}}\right)^{-1} &= \frac{\mathcal{X}_1}{\mathcal{X}_2} \end{aligned}$$

Theorem 2.3.24 *The product of any two $\aleph_{\mathcal{X}}$ numbers yields infinity.*

Proof. It follows from Definition 2.3.14 that

$$\aleph_{\mathcal{X}_1} \cdot \aleph_{\mathcal{X}_2} = \aleph_{(\aleph_{\mathcal{X}_1} \cdot \mathcal{X}_2)} = \aleph_{\left(\aleph_{(\mathcal{X}_1 \mathcal{X}_2)}\right)} \quad .$$

We have $\aleph_{(\mathcal{X}_1 \mathcal{X}_2)} > 1$ so the theorem is proven by Definition 2.3.9.

Theorem 2.3.25 *For any two real numbers $\aleph_{\mathcal{X}_1}$ and $\aleph_{\mathcal{X}_2}$, the quotient $\aleph_{\mathcal{X}_1}/\aleph_{\mathcal{X}_2} = \mathcal{X}_1/\mathcal{X}_2$.*

Proof. Suppose that $\mathcal{X}_1 < \mathcal{X}_2$ and

$$\frac{\aleph_{\mathcal{X}_1}}{\aleph_{\mathcal{X}_2}} = z \quad .$$

We have $\|\aleph_{\mathcal{X}_1}\| < \|\aleph_{\mathcal{X}_2}\|$ so z is guaranteed to have a multiplicative inverse which yields

$$\frac{\aleph_{\mathcal{X}_1}}{\aleph_{(z \cdot \mathcal{X}_2)}} = 1 \quad \Longleftrightarrow \quad \aleph_{\mathcal{X}_1} = \aleph_{(z \cdot \mathcal{X}_2)} \quad \Longleftrightarrow \quad z = \frac{\mathcal{X}_1}{\mathcal{X}_2} \quad .$$

The case of $\mathcal{X}_1 > \mathcal{X}_2$ follows from

$$\left(\frac{\aleph_{\mathcal{X}_1}}{\aleph_{\mathcal{X}_2}}\right)^{-1} = z^{-1} \quad ,$$

and the case of $\mathcal{X}_1 = \mathcal{X}_2$ is trivial.

Remark 2.3.26 The remainder of this section is dedicated to the arithmetic of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ numbers of the form $x = \aleph_{\mathcal{X}} + b$. We have not excluded from this definition values of b larger than any $\mathbb{R}_0$ number, but we have also not yet defined $\mathbb{R}_{\aleph}^0$ numbers which are larger than any $\mathbb{R}_0$ number. Therefore, the following rules may, possibly, only to apply to the case of $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ with $b \in \mathbb{R}_0$. For now, however, we will assume the general case.

Definition 2.3.27 Given $\mathcal{X} > 0$, the arithmetic operations of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ numbers with $x \in \mathbb{R}_0$ are

$$\begin{aligned}
-(\aleph_{\mathcal{X}} + b) &= -\aleph_{\mathcal{X}} - b \\
-(-\aleph_{\mathcal{X}} + b) &= \aleph_{\mathcal{X}} - b \\
\pm(\aleph_{\mathcal{X}} + b) + x = x \pm (\aleph_{\mathcal{X}} + b) &= \pm\aleph_{\mathcal{X}} \pm (b \pm x) \\
\pm(\aleph_{\mathcal{X}} + b) \cdot x = x \cdot \pm(\aleph_{\mathcal{X}} + b) &= \pm(\aleph_{(x \cdot \mathcal{X})} + xb) \\
\frac{\pm(\aleph_{\mathcal{X}} + b)}{x} &= \begin{cases} \text{undefined} & \text{for} \quad x = 0 \\ \pm(\aleph_{(\mathcal{X}/x)} + b/x) & \text{for} \quad x \neq 0 \end{cases} \\
\frac{x}{\pm(\aleph_{\mathcal{X}} + b)} &= 0 \quad .
\end{aligned}$$

Definition 2.3.28 Given $x_1 \in \mathbb{R}_{\aleph}^{\mathcal{X}_1}$, $x_2 \in \mathbb{R}_{\aleph}^{\mathcal{X}_2}$, and $0 < \mathcal{X}_1 \leq \mathcal{X}_2$, the arithmetic operations of $\mathbb{R}_{\aleph}^{\mathcal{X}_1}$ numbers with $\mathbb{R}_{\aleph}^{\mathcal{X}_2}$ numbers are

$$\begin{aligned}
x_1 + x_2 = x_2 + x_1 = (\aleph_{\mathcal{X}_1} + b_1) + (\aleph_{\mathcal{X}_2} + b_2) &= \aleph_{(\mathcal{X}_1 + \mathcal{X}_2)} + (b_1 + b_2) \\
x_2 - x_1 = (\aleph_{\mathcal{X}_2} + b_2) - (\aleph_{\mathcal{X}_1} + b_1) &= \aleph_{(\mathcal{X}_2 - \mathcal{X}_1)} + (b_2 - b_1) \\
x_1 - x_2 &= -(x_2 - x_1) \\
x_1 \cdot x_2 = x_2 \cdot x_1 &= \infty \\
\frac{x_1}{x_2} = \left(\frac{x_2}{x_1}\right)^{-1} &= \begin{cases} c & \text{for} \quad x_1 = cx_2 \\ \text{undefined} & \text{for} \quad \text{other cases} \end{cases} \quad .
\end{aligned}$$

Theorem 2.3.29 *The quotient of $x_1 \in \mathbb{R}_{\aleph}^{\mathcal{X}_1}$ and $x_2 \in \mathbb{R}_{\aleph}^{\mathcal{X}_2}$ is undefined when x_1 and x_2 are not scalar multiples of each other.*

Proof. Suppose

$$\frac{x_1}{x_2} = \frac{\aleph_{\mathcal{X}_1} + b_1}{\aleph_{\mathcal{X}_2} + b_2} = z \quad .$$

Applying the multiplicative inverse of the denominator yields

$$\aleph_{\mathcal{X}_1} + b_1 = \aleph_{(z\mathcal{X}_2)} + zb_2 \quad .$$

The $\aleph_{\mathcal{X}}$ and $\mathbb{R}^0_{\aleph}$ parts of this equation must be separately equal giving

$$\mathcal{X}_1 = z\mathcal{X}_2 \quad , \qquad \text{and} \qquad b_1 = zb_2 \quad .$$

It is impossible to accommodate these two constraints with a single number z unless $x_1 = cx_2$. Therefore, this quotient cannot be defined in the general case.

§2.4 Comparison of Cuts in Lines and Points in Line Segments

In this section, we clarify the cases in which an interior point of a line segment can or cannot be identified with a unique real number, namely the cases $X \equiv x$ and $X \equiv \mathscr{X}$.

Theorem 2.4.1 *If AB is a real line segment of finite length $L \in \mathbb{R}_0$, then every point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. Let $a, b \in \mathbb{R}_0$ and $AB \equiv [a, b]$. The algebraic FDF $\mathcal{D}^\dagger_{AB}(AX)$ is defined to behave exactly as the geometric FDF $\mathcal{D}_{AB}(AX)$ so $\mathcal{D}^\dagger_{AB}(AX)$ must be one-to-one (injective.) By Definition 1.4.13, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

Therefore, the present theorem will be proven if we show that $x_1 = x_2$ for all AB with $L \in \mathbb{R}_0$. To initiate proof by contradiction, assume $x_1, x_2 \in \mathbb{R}_0$ and $x_1 \neq x_2$. Then

$$\min[\mathcal{D}^\dagger_{AB}(AX)] = \frac{\text{len}[a, x_1]}{\text{len}[a, b]} = \frac{x_1 - a}{b - a} \quad ,$$

and

$$\max[\mathcal{D}^\dagger_{AB}(AX)] = \frac{\text{len}[a, x_2]}{\text{len}[a, b]} = \frac{x_2 - a}{b - a} \quad .$$

The one-to-one property of $\mathcal{D}^\dagger_{AB}(AX)$ requires that

$$\frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a} \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad .$$

This contradicts the assumption $x_1 \neq x_2$.

Theorem 2.4.2 *If AB is a real line segment of infinite length $L = \infty$, then no point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. By Definition 1.4.13, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

The one-to-one property of $\mathcal{D}^\dagger_{AB}(AX)$ requires that

$$f(a) = f(b) \quad \Longleftrightarrow \quad a = b \quad .$$

For $\mathcal{D}^\dagger_{\mathbf{AB}}(AX)$ we have

$$\min[\mathcal{D}^\dagger_{\mathbf{AB}}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \infty]} = \frac{x_1}{\infty} = 0 \quad ,$$

and

$$\max[\mathcal{D}^\dagger_{\mathbf{AB}}(AX)] = \frac{\text{len}[0, x_2]}{\text{len}[0, \infty]} = \frac{x_2}{\infty} = 0 \quad .$$

We find that

$$0 = 0 \quad \not\Longleftrightarrow \quad x_1 = x_2 \quad .$$

Any two $x_1, x_2 \in \mathbb{R}_0$ will satisfy $\mathcal{D}^\dagger_{\mathbf{AB}}(AX) = \mathcal{D}^\dagger_{\mathbf{AB}}(AX)$. Therefore, no point X in a real line segment of infinite length has a unique algebraic representation as one and only one real number. ∅

Example 2.4.3 If we separate an endpoint from an algebraic interval then we may write

$$[a, b] = \{a\} \cup (a, b] \quad .$$

To separate an endpoint from a line segment we write

$$AB = A + AB \quad .$$

If A has an algebraic representation $\mathscr{A}$ such that $\text{len}\,\mathscr{A} > 0$, then the only way that we can leave the length of AB unchanged after removing A is for AB to have infinite length. Given $\text{len}\,\mathscr{A} > 0$

$$\|AB\| - \text{len}\,\mathscr{A} = \|AB\| \quad \Longleftrightarrow \quad \|AB\| = \infty \quad .$$

Remark 2.4.4 Theorems 2.4.1 and 2.4.2 do not cover all cases of L. We have not considered two remaining cases of finite L. The lesser case is $L \in \mathbb{R}^0_\aleph \setminus \mathbb{R}_0$. The greater case is finite length $L \in \mathbb{R}_\infty$. Since we have not yet introduced numbers through which to describe the lesser case, we will withhold the result regarding the multivaluedness of points in such line segments.

Theorem 2.4.5 *If AB is a real line segment with finite length $L \in \mathbb{R}_\infty$, then no point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. Let $AB \equiv [0, \aleph_\mathcal{X} + b]$. We cover every $L \in \mathbb{R}_\infty$ with $0 < \mathcal{X} \leq 1$. By Definition 1.4.13, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

The one-to-one property of $\mathcal{D}^\dagger_{AB}(AX)$ requires that

$$f(x_1) = f(x_2) \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad .$$

For $\mathcal{D}^\dagger_{AB}(AX)$ we have

$$\min[\mathcal{D}^\dagger_{AB}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \aleph_\mathcal{X} + b]} = \frac{x_1}{\aleph_\mathcal{X} + b} = 0 \quad ,$$

and

$$\max[\mathcal{D}^\dagger_{AB}(AX)] = \frac{\text{len}[0, x_2]}{\text{len}[0, \aleph_\mathcal{X} + b]} = \frac{x_2}{\aleph_\mathcal{X} + b} = 0 \quad ,$$

We find that

$$0 = 0 \qquad \not\Longleftrightarrow \qquad x_1 = x_2 \quad .$$

Any two $x_1, x_2 \in \mathbb{R}_0$ will satisfy $\mathcal{D}^\dagger_{AB}(AX) = \mathcal{D}^\dagger_{AB}(AX)$. Therefore, no point X in a real line segment of finite length $L \in \mathbb{R}_\infty$ has a unique algebraic representation as one and only one real number. ∎

§3 The Neighborhood of Infinity

§3.1 Algebraic Infinity

In Section 2.1, we have shown that neither the algebraic FDF of the first kind nor the second has the analytic form form of $\mathcal{D}^\dagger_{AB}$. The second kind was ruled out by Theorem 2.1.19 when we showed that $\mathcal{D}''_{AB}$ is not one-to-one. Furthermore, $\mathcal{D}'_{AB}$ was provisionally eliminated based on an unallowable discontinuity at infinity. In Theorem 2.1.24, we showed that $\mathcal{D}'_{AB}$ cannot conform to the

Cauchy criterion for continuity at infinity although this is a requirement for $\mathcal{D}^{\dagger}_{AB} = \mathcal{D}_{AB}$. Specifically, Theorem 2.1.24 contains a requirement

$$|x - \infty| < \delta \qquad \Longleftrightarrow \qquad \delta > \infty \quad ,$$

but there is no such δ. What is the source of this discrepancy?

We are perfectly well motivated to say, "Let there be an algebraic FDF which totally replicates the behavior of the geometric FDF and also allows us to compute the fractional distance as a numerical ratio." Indeed, it is highly preferable that $\mathcal{D}^{\dagger}_{AB}$ be either $\mathcal{D}'_{AB}$ or $\mathcal{D}''_{AB}$ because, seemingly, anything else would be contrived. Since it the properties assigned to ∞ which cause the disagreement between $\mathcal{D}^{\dagger}_{AB}$ and $\mathcal{D}'_{AB}$, and those properties are by no means set in stone, we should examine those properties with the intention to generate the correct behavior for $\mathcal{D}'_{AB}$ at infinity through revisions. In Section 3.3, we will do exactly that. The present section develops the properties of infinity which will facilitate the desired behavior.

Much of this section originally appeared in Reference [1] wherein we suppressed the additive absorptive property of ∞ by putting a hat on it, $\widehat{\infty}$, to generate a rule regarding the operations. This section extends what was developed in Reference [1] so that the hat will suppress all absorptive properties, not only the additive absorptive property. While this section stands alone, the reader is encouraged to consult Reference [1] for somewhat more information regarding $\widehat{\infty}$.

Definition 3.1.1 ∞ is a number called geometric infinity.

Definition 3.1.2 $\widehat{\infty}$ is a number called algebraic infinity.

Definition 3.1.3 Additive absorption is a property of $\pm\infty$ such that all $x \in \mathbb{R}$ are additive identities of $\pm\infty$. The additive absorptive property is

$$\pm\infty \pm x = \pm\infty \mp x = \pm\infty \quad .$$

Definition 3.1.4 Multiplicative absorption is a property of $\pm\infty$ such that all non-zero $x \in \mathbb{R}$ are multiplicative identities of $\pm\infty$. The multiplicative absorptive property is

$$\pm\infty \cdot x = \pm x \cdot \infty = \begin{cases} \pm\infty & \text{for} \quad x > 0 \\ \mp\infty & \text{for} \quad x < 0 \end{cases} .$$

Definition 3.1.5 Algebraic infinity $\widehat{\infty}$ is not vested with the properties of additive and/or multiplicative absorption.

Definition 3.1.6 If $\mathcal{X} \geq 1$, then $\aleph_{\mathcal{X}} = \widehat{\infty}$. (This definition supersedes Definition 2.3.9.)

Example 3.1.7 In Definition 2.3.14, we have given an arithmetic operation for $0 < \mathcal{X} < 1$:

$$\mathcal{X}^{-1}\aleph_{\mathcal{X}} = \infty \quad .$$

Since $\mathcal{X}^{-1}$ has a multiplicative inverse, consistency requires that

$$\mathcal{X}\infty = \mathcal{X}\mathcal{X}^{-1}\aleph_{\mathcal{X}} = \aleph_{\mathcal{X}} = \infty \quad , \qquad \text{but} \qquad \aleph_{\mathcal{X}} \neq \infty$$

This contradiction is avoided with $\widehat{\infty}$:

$$\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} = \mathcal{X}\aleph_1 = \aleph_{\mathcal{X}} \quad .$$

This example shows that $\widehat{\infty}$ can be used to avoid any contradictions of the form suggested in Remark 2.3.15.

Definition 3.1.8 Algebraic and geometric infinity both describe the same affinely extended real number:

$$\lim_{n\to\infty}\sum_{k=1}^{n} k = \widehat{\infty} \quad , \qquad \text{and} \qquad \lim_{n\to\infty}\sum_{k=1}^{n} k = \infty \quad .$$

Therefore,

$$\pm\widehat{\infty} = \pm\infty \qquad \Longrightarrow \qquad \mathbb{R} \equiv (-\widehat{\infty}, \widehat{\infty}) \quad .$$

Remark 3.1.9 It is sometimes claimed, without proof, that one cannot place endpoints at the ends of $\mathbb{R} \equiv (-\infty, \infty)$ because the notion of an endpoint contradicts the notion of the infinite geometric extent of a 1D space extending infinitely far in both directions. Although this claim is weak and does not require a workaround at all because the soft brackets on $(-\infty, \infty)$ may be replaced with square brackets obviously, the purpose of this remark is to develop a workaround based on a distinction between algebraic and geometric infinity.

Geometric infinity shall be a number which cannot be included as an endpoint without contradicting the notion of the infinite geometric extent of a number line. Definition 1.2.2 defines a number line as a 1D metric space with the Euclidean metric

$$L(x, y) = |y - x| \quad .$$

To make the distinction with algebraic infinity, first let

$$AB \equiv [-\pi/2, \pi/2] \quad , \qquad \text{and} \qquad x \in [-\pi/2, \pi/2] \quad .$$

x is a chart on AB that varies between $\pm\pi/2$. To show how algebraic infinity differs from geometric infinity, define a second chart on AB such that

$$x' = \tan(x) \quad .$$

Expressing AB as an algebraic interval in terms of x' yields

$$AB \equiv [-\widehat{\infty}, \widehat{\infty}] \quad , \qquad \text{and} \qquad x' \in [-\widehat{\infty}, \widehat{\infty}] \quad .$$

It is unquestionable that the points A and B exist and are well-defined in the x chart, and it is not possible to disrupt the geometric configuration by introducing a second chart onto AB. Furthermore, when we introduce the Euclidean metric

$$L'(x', y') = |y' - x'| \quad ,$$

onto AB, we have completely reconstructed all of $\mathbb{R}$ on the interior of AB. Since AB is a line segment that includes its endpoints by definition, here we have added endpoints to the end of the real line without violating the geometric notion of infinite extent which is imbued to $\mathbb{R}$. The main difference between algebraic and geometric infinity is that the former can be embedded in a larger space but geometric infinity is totally maximal and cannot be embedded in something larger than itself.

By making this remark, we have not strictly required any specific behavior. This workaround is not needed at all because the argument it works around is not sound. We have simply offered the loose framework of an argument which can be used to invalidate the unsubstantiated claim that it is not possible to put endpoints at infinity.

Remark 3.1.10 Consider $\mathbf{AB}$. Although the intervals

$$AB \equiv [0, \pi/2] \quad , \qquad \text{and} \qquad AB \equiv [0, \widehat{\infty}] \quad ,$$

are the exact same line segment expressed in two conformally related charts, x and x', notice how the conformal relationship $x' = \tan(x)$ changes the density of numbers along AB. We have shown with the fractional distance functions that all of $\mathbb{R}_0$ is compactified onto the left endpoint A of $\mathbf{AB}$ and that all other points along $\mathbf{AB}$ have algebraic representations as numbers in the neighborhood of infinity: $x \in \mathbb{R}_\infty$. However, the tangent of any $x \in [0, \pi/2)$ is an $\mathbb{R}_0$ number meaning that all of $\mathbb{R}_\infty$ is compactified onto the right endpoint B of $\mathbf{AB}$. By switching to the $L'(x', y')$ metric, we restore the agreement about which neighborhood is compactified onto which endpoint, but is notable that we have the freedom to stretch and moosh $\mathbb{R}_0$ and $\mathbb{R}_\infty$ as desired.

Similarly, consider the Riemann sphere. The complex plane is mapped onto $\mathbb{S}^2$ via projection of the polar ray. The usual construction results in all of $\mathbb{R}_\infty$ being compacted onto the polar point at the maximum of the zenith. However, if we chose a sphere whose radius is larger than any $\mathbb{R}_0$ number then the polar ray would paint a continuum of $\aleph_\mathcal{X}$ numbers onto a great circle of $\mathbb{S}^2$ while $\mathbb{R}_0$ would be compacted onto a point. Such compactifications as mentioned in this remark are known as conformal transformations.

Definition 3.1.11 $\widehat{\infty}$ is endowed with the non-contradiction property. This property requires that the hat which differentiates algebraic infinity $\pm\widehat{\infty}$ from canonical infinity $\pm\infty$ is inserted and removed by choice except in the case where the hat invokes a contradiction and must be removed by definition. If a contradiction is obtained via non-absorptivity, then the hat must be removed to alleviate the contradiction, thus the name of the non-contradiction property.

Example 3.1.12 Examples in which the hat does not invoke a contradiction and may be left in place are

$$x = \widehat{\infty} - b \quad , \qquad \text{and} \qquad \aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad .$$

Example 3.1.13 An example of a statement in which the hat invokes a contradiction and may not be left in place is given by two sequences

$$x_n = \sum_{k=1}^{n} k \quad , \qquad \text{and} \qquad y_n = c_0 + \sum_{k=1}^{n} k \quad ,$$

where $n \in \mathbb{N}$ and c_0 is some non-zero real number. Since ∞ and $\widehat{\infty}$ are the same number, we can use Definitions 1.3.2 and 3.1.5 to write

$$\lim_{n\to\infty} x_n = \infty = \widehat{\infty} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty = \widehat{\infty} \quad .$$

We may also write, however,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} c_0 + \lim_{n\to\infty} x_n = c_0 + \widehat{\infty} \quad .$$

This delivers an equality

$$\widehat{\infty} = c_0 + \widehat{\infty} \quad ,$$

which contradicts the non-absorption of $\widehat{\infty}$. At this point, we must cease to delay additive absorption by removing the hat. Then

$$\infty = c_0 + \infty \quad ,$$

demonstrates the usual additive absorptive property of infinity and there is no contradiction. Since we have invoked the contradiction via the non-absorptive property, the non-contradiction property requires that we remove the hat. In any case, a very large arena of mathematical structure may be explored via the non-absorptive property without invoking any contradictions.

Remark 3.1.14 When the $\pm\infty$ symbol appears as $\pm\widehat{\infty}$, consider the hat to be an instruction to delay the absorptive operations of $\pm\infty$ indefinitely or until such delay causes a contradiction. The instruction to "delay absorption" should be understood to mean that absorption is not a property of $\pm\widehat{\infty}$ but that the absorptive properties can be trivially implemented after an *ad hoc* decision to remove the hat by choice or after its removal is required by the non-contradiction property of Definition 3.1.11. Although the interval $[-\widehat{\infty},\widehat{\infty}]$ can be conformally constructed on any subset of $(-\infty,\infty)$, when we remove the hat from $\widehat{\infty}$ we preclude the existence of any points to the right of the point we have associated with $\widehat{\infty}$. When we remove the hat, this point becomes geometric infinity.

Definition 3.1.15 $\widehat{\infty}$ is such that for any non-zero $b\in\mathbb{R}_0$

$$\begin{aligned} \pm\widehat{\infty}+b&=b\pm\widehat{\infty} \\ \pm\widehat{\infty}-b&=-b\pm\widehat{\infty} \\ \pm\widehat{\infty}+\big(-b\big)&=\pm\widehat{\infty}-b \\ \pm\widehat{\infty}+b&=\pm\widehat{\infty}-\big(-b\big) \\ -\big(\pm\widehat{\infty}\big)&=\mp\widehat{\infty} \\ \pm\widehat{\infty}\cdot b=b\cdot\pm\widehat{\infty}&=\pm\aleph_b \\ \frac{\pm\widehat{\infty}}{b}&=\pm\aleph_{(b^{-1})} \\ \frac{b}{\pm\widehat{\infty}}&=0 \quad . \end{aligned}$$

Definition 3.1.16 $\widehat{\infty}$ is such that

$$\begin{aligned} \pm\widehat{\infty}+0=0\pm\widehat{\infty}=\pm\widehat{\infty}-0&=\text{undefined} \\ \pm\widehat{\infty}\cdot 0=0\cdot\pm\widehat{\infty}&=\text{undefined} \\ \frac{\pm\widehat{\infty}}{0}&=\text{undefined} \\ \frac{0}{\pm\widehat{\infty}}&=0 \quad . \end{aligned}$$

Remark 3.1.17 The non-existence of an additive identity element for algebraic infinity is treated in Reference [1].

Definition 3.1.18 The arithmetic operations of $\widehat{\infty}$ numbers with $\aleph_{\mathcal{X}}$ numbers having $0<\mathcal{X}<1$ are

$$\begin{aligned} -(\pm\widehat{\infty}+\aleph_{\mathcal{X}})&=\mp\widehat{\infty}-\aleph_{\mathcal{X}} \\ \widehat{\infty}\pm\aleph_{\mathcal{X}}=\pm\aleph_{\mathcal{X}}+\widehat{\infty}&=\aleph_{(1\pm\mathcal{X})} \\ -\widehat{\infty}\pm\aleph_{\mathcal{X}}=\pm\aleph_{\mathcal{X}}-\widehat{\infty}&=-\aleph_{(1\mp\mathcal{X})} \\ \pm\widehat{\infty}\cdot\aleph_{\mathcal{X}}=\aleph_{\mathcal{X}}\cdot(\pm\widehat{\infty})&=\pm\widehat{\infty} \end{aligned}$$

$$\frac{\aleph_{\mathcal{X}}}{\widehat{\infty}} = \mathcal{X}$$
$$\frac{\widehat{\infty}}{\aleph_{\mathcal{X}}} = \frac{1}{\mathcal{X}} \quad .$$

Example 3.1.19 This example demonstrates an aspect of consistency in the arithmetic operations of $\widehat{\infty}$ and $\aleph_{\mathcal{X}}$. Notice that

$$\mathcal{X} = \frac{\aleph_{\mathcal{X}}}{\widehat{\infty}} = \frac{\aleph_{\mathcal{X}}}{\widehat{\infty}} \cdot \frac{\aleph_{\mathcal{X}}^{-1}}{\aleph_{\mathcal{X}}^{-1}} = \frac{1}{\left(\dfrac{\widehat{\infty}}{\aleph_{\mathcal{X}}}\right)} = \frac{1}{\left(\dfrac{1}{\mathcal{X}}\right)} = \mathcal{X} \quad .$$

Example 3.1.20 It is a well known property of limits that

$$\lim_{x\to\infty} x = \lim_{x\to\infty} c_0 + x = \infty \quad .$$

If we take $c_0 = \aleph_{\mathcal{X}}$, then

$$\lim_{x\to\infty} x = \lim_{x\to\infty} \aleph_{\mathcal{X}} + x = \infty \quad ,$$

but a fine point must be made between

$$\lim_{x\to-\infty} c_0 + x = -\infty \quad , \qquad \text{and} \qquad \lim_{x\to-\infty} \aleph_{\mathcal{X}} + x = \aleph_{\mathcal{X}} - \infty \quad .$$

One might apply the arithmetic of $\widehat{\infty}$ and $\aleph_{\mathcal{X}}$ (Definition 3.1.18) to write

$$\aleph_{\mathcal{X}} - \infty = -\aleph_{(1-\mathcal{X})} \neq -\infty \quad .$$

Upon closer inspection, however, we see that infinity appears in this expression unhatted. Unhatted infinity (geometric infinity) has the property of additive absorption so

$$\aleph_{\mathcal{X}} - \infty = -\infty \neq -\aleph_{(1-\mathcal{X})} \quad .$$

Remark 3.1.21 Notice that $\lim_{x\to\infty} x = \infty$ and yet we have the property of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ numbers that the b in $x = \aleph_{\mathcal{X}_1} + b$ can seemingly increase forever without x reaching $\aleph_{\mathcal{X}_2} \neq \aleph_{\mathcal{X}_1}$. After we develop a some paradoxes in Section 4, we will propose a solution which will address the point raised in this remark.

§3.2 Greatest Real Numbers in the Neighborhood of Infinity

The main purpose of this section is to treat the properties of real numbers $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ for the special case of $\mathcal{X} = 1$.

Definition 3.2.1 The δ-neighborhood of a number $x \in \mathbb{R}$ is a a set $(x-\delta, x+\delta)$ or some permutation thereof including endpoints.

Definition 3.2.2 The δ-neighborhood of a point $X \in AB$ is a line segment YZ where

$$|\mathcal{D}_{AB}(AX) - \mathcal{D}_{AB}(AY)| = |\mathcal{D}_{AB}(AX) - \mathcal{D}_{AB}(AZ)| = \delta \quad .$$

Remark 3.2.3 Without regard to the δ-neighborhood of any point or number, in the present study we define neighborhoods with the geometric FDF, as in Definitions 2.3.5 and 2.3.6. If $\mathcal{D}_{\mathbf{AB}}(AX) = 0$, then the numbers in the algebraic representation of X are said to be in the neighborhood of the origin. They are said to be in the neighborhood of infinity otherwise. Neither of these neighborhoods, neither that of the origin nor that of infinity, are defined as δ-neighborhoods.

Definition 3.2.4 $\widehat{\mathbb{R}}$ is a subset of all real numbers

$$\widehat{\mathbb{R}} = \{\pm(\widehat{\infty} - b) \mid b \in \mathbb{R}_0 \ , \ b > 0\} \quad .$$

Positive numbers in $\widehat{\mathbb{R}}$ are denoted $\widehat{\mathbb{R}}^+$. They are said to be the greatest real numbers and they have the property $\widehat{\mathbb{R}}^+ \subset \mathbb{R}^1_{\aleph}$.

Theorem 3.2.5 *If $x \in \widehat{\mathbb{R}}^+$ and $x \in X$, then $\mathcal{D}_{\mathbf{AB}}(AX) = 1$.*

Proof. If $x \in \widehat{\mathbb{R}}^+$, then $x \in \mathbb{R}^1_{\aleph}$ and proof follows trivially.

Remark 3.2.6 Note that

$$\widehat{\mathbb{R}}^+ \subset \mathbb{R}^1_{\aleph} \ , \quad \widehat{\mathbb{R}}^+ \subset \mathbb{R}_\infty \ , \qquad \text{but} \qquad \mathbb{R}^1_{\aleph} \not\subset \mathbb{R}_\infty \quad .$$

$\mathbb{R}^1_{\aleph}$ cannot be a subset of $\mathbb{R}$ because numbers of the form $\aleph_1 + b = \widehat{\infty} + b$ are not real when $b \geq 0$.

Theorem 3.2.7 *All numbers $x \in \widehat{\mathbb{R}}$ are cuts in the affinely extended real number line, i.e.: they are affinely extended real numbers.*

Proof. By Definition 1.3.6, an affinely extended real number is a cut in, or an endpoint of, the affinely extended real number line. Definition 1.2.5 requires that a cut separates one line into two pieces. Noting that

$$x \in \widehat{\mathbb{R}} \quad \Longrightarrow \quad x = \pm\big(\widehat{\infty} - b\big) \quad ,$$

observe that

$$\begin{aligned}[-\infty,\infty] &= [-\infty,\widehat{\infty}-b)\cup[\widehat{\infty}-b,\infty]\\ [-\infty,\infty] &= [-\infty,-\widehat{\infty}+b)\cup[-\widehat{\infty}+b,\infty] \quad .\end{aligned}$$

All numbers $x\in\widehat{\mathbb{R}}$ conform to the definition of affinely extended real numbers.

Theorem 3.2.8 *All numbers $x\in\widehat{\mathbb{R}}$ are real numbers.*

Proof. If a number is an affinely extended real number $x\in\overline{\mathbb{R}}$ and $x\neq\pm\infty$, then by Main Theorem 1.3.7 we have $x\in\mathbb{R}$. In the absence of additive absorption

$$\pm(\widehat{\infty}-b)\neq\pm\widehat{\infty}=\pm\infty \quad ,$$

because it is the definition of $\widehat{\mathbb{R}}$ that $b\neq 0$. Also note that

$$\begin{aligned}(-\infty,\infty) &= (-\infty,\widehat{\infty}-b)\cup[\widehat{\infty}-b,\infty)\\ (-\infty,\infty) &= (-\infty,-\widehat{\infty}+b)\cup[-\widehat{\infty}+b,\infty) \quad .\end{aligned}$$

All numbers $x\in\widehat{\mathbb{R}}$ satisfy Definition 1.2.6.

Definition 3.2.9 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\mathbb{R}_0$ numbers are

$$\begin{aligned}-(\widehat{\infty}-b) &= -\widehat{\infty}+b\\ -(-\widehat{\infty}+b) &= \widehat{\infty}-b\\ \pm(\widehat{\infty}-b)+x=x\pm(\widehat{\infty}-b) &= \begin{cases}\pm\widehat{\infty}\mp(b-x) & \text{if}\quad b\neq x\\ \pm\widehat{\infty} & \text{if}\quad b=x\end{cases}\\ \pm(\widehat{\infty}-b)\cdot x=x\cdot\pm(\widehat{\infty}-b) &= \begin{cases}\pm(\aleph_x-xb) & \text{if}\quad x\neq 0\\ \text{undefined} & \text{if}\quad x=0\end{cases}\\ \frac{\pm(\widehat{\infty}-b)}{x} &= \begin{cases}\pm\aleph_{(x^{-1})}\mp\dfrac{b}{x} & \text{if}\quad x\neq 0\\ \text{undefined} & \text{if}\quad x=0\end{cases}\\ \frac{x}{\pm(\widehat{\infty}-b)} &= 0 \quad .\end{aligned}$$

Theorem 3.2.10 *The quotient of a number $x\in\mathbb{R}_0$ divided by a number $y\in\widehat{\mathbb{R}}$ is identically zero.*

Proof. Let z be any non-zero real number such that

$$\frac{x}{y} = z \quad .$$

Since $||x|| < ||y||$, we have $||z|| < 1$ which implies $z \in \mathbb{R}_0$. All non-zero $\mathbb{R}_0$ numbers have a multiplicative inverse. We find, therefore, that

$$\frac{x}{zy} = 1 \qquad \Longleftrightarrow \qquad x = zy \quad .$$

The hat on $\widehat{\infty}$ suppresses absorption so

$$zy = z \cdot \pm(\widehat{\infty} - b) = \pm(\aleph_z - zb) \quad .$$

This delivers a contradiction because $z \neq 0$ requires that x is a real number in the neighborhood of infinity while we have already defined x to be a real number in the neighborhood of the origin. Therefore, the only possible numerical value for x/y is 0.

Definition 3.2.11 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\widehat{\mathbb{R}}$ numbers are

$$\begin{aligned}
\pm(\widehat{\infty} - b) \pm (\widehat{\infty} - a) &= \pm\widehat{\infty} \mp (b + a) \\
\pm(\widehat{\infty} - b) \mp (\widehat{\infty} - a) &= \pm(a - b) \\
\pm(\widehat{\infty} - b)(\widehat{\infty} - a) &= \pm\widehat{\infty} \\
\frac{\widehat{\infty} - b}{\widehat{\infty} - a} &= \text{undefined} \quad .
\end{aligned}$$

Theorem 3.2.12 *Quotients of the form $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ are undefined.*

Proof. Observe that

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \frac{\widehat{\infty}}{\widehat{\infty} - a} - \frac{b}{\widehat{\infty} - a} \quad .$$

Insert the multiplicative identity into the first term so that

$$\frac{\widehat{\infty} \cdot 1}{\widehat{\infty} - a} = \widehat{\infty} \cdot \left(\frac{1}{\widehat{\infty} - a}\right) = \widehat{\infty} \cdot 0 = \text{undefined} \quad .$$

Remark 3.2.13 Note that the quotient

$$x = \frac{\widehat{\infty} - b}{\widehat{\infty} - a} \quad .$$

cannot depend on b. Therefore, if defined, this quotient would have the same value regardless of whether or not the numerator was greater than or less than the denominator.

§3.3 Continuity of $\mathcal{D}^\dagger_{AB}(AX)$ Revisited

In Theorem 2.1.22 we showed that the properties usually attributed to infinity break the agreement of $\mathcal{D}^\dagger_{AB}$ and $\mathcal{D}'_{AB}$ in the discontinuity of the latter at infinity. Since $\mathcal{D}_{AB}$ and $\mathcal{D}^\dagger_{AB}$ are continuous everywhere on an arbitrary line segment AB, Theorem 2.1.22 hints that the properties usually assigned to infinity are wrong. One expects that the correct properties of infinity would allow us to use either of $\mathcal{D}'_{AB}$ or $\mathcal{D}''_{AB}$ for $\mathcal{D}^\dagger_{AB}$. In this section, we will show that the properties assigned to $\widehat{\infty}$ support the agreement of $\mathcal{D}^\dagger_{AB}$ with $\mathcal{D}'_{AB}$. In this section, we derive a remarkable result showing that the ordinary Cauchy definition of a limit can be used to compute a limit at infinity. This is commonly presumed to be impossible.

Main Theorem 3.3.1 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{\mathbf{AB}}(AX)$ converges to 1 at $B = \widehat{\infty}$.*

Proof. According to the Cauchy definition of the limit of $f(x) : D \to R$ at infinity, we say that

$$\lim_{x\to\widehat{\infty}} f(x) = L \quad ,$$

if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \qquad \text{s.t} \qquad \forall x \in D \quad ,$$

we have

$$0 < |x - \widehat{\infty}| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \varepsilon \quad .$$

In Theorem 2.1.24, we attempted to show this limit in the approach to geometric infinity $x \to \infty$. At that point, we had to stop because there is no $\delta \in \mathbb{R}$ such that $\infty - x < \delta$. Now we may choose $x \in \mathbb{R}_\infty$ and use

$$|(\widehat{\infty} - b) - \widehat{\infty}| = b \quad , \qquad \text{or} \qquad |\aleph_{\mathcal{X}} - \widehat{\infty}| = \aleph_{(1-\mathcal{X})} \quad ,$$

to follow the usual prescription for the Cauchy definition of a limit, **even at infinity**. To that end, let $\delta = \aleph_{(\epsilon/2)}$. Then the Cauchy definition requires that

$$0 < |x - \widehat{\infty}| < \aleph_{(\epsilon/2)} \quad \text{and} \quad |\mathcal{D}'_{\mathbf{AB}}(AX) - \mathcal{D}'_{\mathbf{AB}}(AB)| < \epsilon \quad .$$

Evaluation of the δ expression yields

$$\widehat{\infty} - x < \aleph_{(\epsilon/2)} \qquad \Longleftrightarrow \qquad x > \aleph_{\left(1-\frac{\epsilon}{2}\right)} \quad .$$

Evaluation of the ϵ expression yields

$$\left|\frac{\text{len}[0,x]}{\text{len}[0,\widehat{\infty}]} - 1\right| = \left|\frac{x}{\widehat{\infty}} - 1\right| < \left|\frac{\aleph_{\left(1-\frac{\epsilon}{2}\right)}}{\widehat{\infty}} - 1\right| = \left|\left(1 - \frac{\epsilon}{2}\right) - 1\right| = \left|-\frac{\epsilon}{2}\right| < \epsilon \quad .$$

Therefore,

$$\lim_{x\to\widehat{\infty}} \mathcal{D}'_{\mathbf{AB}}(AX) = 1 \quad .$$

This limit demonstrates the continuity of $\mathcal{D}'_{\mathbf{AB}}$ at infinity.

Conjecture 3.3.2 The algebraic fractional distance function $\mathcal{D}^{\dagger}_{AB}$ is an algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$.

§3.4 The Archimedes Property of Real Numbers

In this section, we make several different statements of the Archimedes property of real numbers and compare them. When this property is defined directly by the literal quote which appeared in Euclid's Elements, there are certain unfavorable and/or pathological issues such as negative numbers not having the Archimedes property. As a result, modern statements are introduced to guarantee that all non-zero real numbers have this property. Most generally, the Archimedes property states that there is no greatest (or least) real number. There exist many different ways to phrase this condition in precise mathematical language and the purpose of this section is to examine some of them closely.

Definition 3.4.1 The statement of the Archimedes property which appears in Euclid's Elements, and which was attributed by Archimedes to his predecessor Eudoxus, *and* which is very often taken to be the definitive statement of the Archimedes property of real numbers, is this:

> "Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another."

This statement is referred to hereafter as the Euclid statement.

Definition 3.4.2 The provisional mathematical statement of the Euclid statement is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

Remark 3.4.3 The above statement is called provisional pending some clarifications about numbers larger than any natural number.

Definition 3.4.4 The main gist of the Archimedes property of real numbers is:

> "For every real number, there exists a greater real number."

This statement is referred to hereafter as the general statement.

Definition 3.4.5 The mathematical statement of the general statement is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \quad .$$

Remark 3.4.6 The mathematical statement of the general statement (Definition 3.4.5) differs from the provisional mathematical statement of the Euclid statement (Definition 3.4.2) by not restricting the multiplier to $n \in \mathbb{N}$. Although the Euclid statement retains all of its meaning when the multiplier is allowed to be any real number, for some reason the multiplier is often defined as a natural number. Although nothing is added to the Archimedes property by such a restriction, this restriction is quite common. It is the opinion of this writer, however, that a property of real numbers is best stated in terms of real numbers alone whenever possible. For this reason, we have stated the Archimedes property without this restriction in Definition 3.4.5. What we call "the Euclid statement" restricts the multiplier to $\mathbb{N}$ even though Euclid makes no such explicit restriction in his book. We call the $\mathbb{N}$-restricted Archimedes property the Euclid statement only because it so commonly phrased in terms of a natural multiplier. However, what we have called "the general statement" totally conforms in every way to the Archimedes property as it appears in Euclid's Elements. The remainder of this section demonstrates the superiority of the unrestricted general statement over the needlessly constrained Euclid statement.

Theorem 3.4.7 *The real number zero obeys neither the provisional Euclid statement nor the general statement.*

Proof. Suppose that $n \in \mathbb{N}$, $y \in \mathbb{R}$, and $y > 0$. Observe that

$$0 < y \quad \Longrightarrow \quad 0n \not> y \quad .$$

This demonstrates the nonconformity of $x = 0$ to the provisional Euclid statement. The nonconformity to the general statement is demonstrated with a multiplier $z \in \mathbb{R}$:

$$0 < y \quad \Longrightarrow \quad 0z \not> y \quad .$$

Theorem 3.4.8 *Negative $\mathbb{R}_0$ numbers do not obey the provisional Euclid statement.*

Proof. The provisional Euclid statement is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

Supposing that $x < 0$, $x \in \mathbb{R}_0$, $y \in \mathbb{R}$, and $n \in \mathbb{N}$, observe that

$$x < 0 \quad \Longrightarrow \quad nx < x \quad , \qquad \text{but} \qquad x < y \quad \Longrightarrow \quad nx \not> y \quad .$$

If x is a negative number, then multiplying it by a natural number decreases its magnitude. ✐

Remark 3.4.9 On account of of the failure of negative $\mathbb{R}_0$ numbers to conform to the provisional Euclid statement, it is often claimed that Euclid's archaic term "magnitudes" refers only to that which can be physically measured. It is said, therefore, that the Euclid statement appears in the context of positive real numbers only. This caveat about real numbers being strictly positive does not appear in anything ever written by Euclid, Archimedes, or Eudoxus. Even if it did appear explicitly, it would be rejected by those who prefer to define the Archimedes property as something that negative real numbers have. The caveat that magnitudes only refer to positive numbers is added by modern mathematicians because it is preferable to define the Archimedes property of real numbers in a way that makes all non-zero real numbers have it. However, as we will show in Theorem 3.4.10, the provisional mathematical statement of the Euclid statement (Definition 3.4.2) is insufficient even to cover all positive numbers. For the statement to be robust, it must be clarified that magnitudes are strictly positive *and* it must be further clarified that Archimedean magnitudes are strictly in the neighborhood of the origin.

Theorem 3.4.10 *Positive real numbers do not always obey the provisional Euclid statement.*

Proof. The provisional Euclid statement is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

Suppose $x > 0$, $x \in \mathbb{R}_0$, $y \in \mathbb{R}_{\aleph}^{\mathcal{X}}$, and $0 < \mathcal{X} < 1$. It follows that

$$x < y \quad .$$

However, for any $n \in \mathbb{N}$ we have

$$nx \in \mathbb{R}_0 \quad \Longrightarrow \quad nx \not> y \quad .$$ ✐

Definition 3.4.11 The exact mathematical statement of the Euclid statement is

$$\forall x, y \in \mathbb{R}_0 \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

Remark 3.4.12 The exact statement of the Euclid statement builds on the requirement that x and y are positive by further requiring that they are in $\mathbb{R}_0$. This restriction makes the exact statement robust where the provisional statement fails in certain cases of x and y not $\mathbb{R}_0$. Again, here we take no liberties beyond the liberties ordinarily taken: it is as obvious that Euclid was writing about numbers near the origin as it is obvious that he was writing about positive numbers.

Theorem 3.4.13 *Positive $\widehat{\mathbb{R}}$ numbers behave exactly like negative $\mathbb{R}_0$ numbers under the Euclid statement.*

Proof. We have shown in Theorem 3.4.8 that negative numbers decrease in magnitude when multiplied by a natural number and the present theorem will be proven if we demonstrate the same for positive $\widehat{\mathbb{R}}$ numbers. Suppose $x \in \widehat{\mathbb{R}}$ and $x = (\widehat{\infty} - b)$. For $n \in \mathbb{N}$ we have

$$n(\widehat{\infty} - b) = \widehat{\infty} - nb \qquad \Longrightarrow \qquad nx < x \quad .$$

(Here $n \cdot \widehat{\infty} = \widehat{\infty}$ because $n \geq 1$.) Multiplication by a natural number lessens positive $\widehat{\mathbb{R}}$ numbers so multiplication by n can never start with $x < y$ and then yield $nx > y$. As with negative numbers, multiplication by n decreases the magnitude of positive $\widehat{\mathbb{R}}$ numbers.

Theorem 3.4.14 *Not all $\mathbb{R}_{\aleph}^{\mathcal{X}}$ numbers obey the provisional Euclid statement.*

Proof. Suppose $x < y$ and $x = \aleph_{\mathcal{X}} + b$. By the arithmetic of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ (Definition 2.3.27), we have

$$n(\aleph_{\mathcal{X}} + b) = \aleph_{(n\mathcal{X})} + nb \quad .$$

The Euclid statement requires $x < y$ and $nx > y$. However, if $\mathcal{X} \geq 0.5$ and $b > 0$ then nx is not a real number.

Remark 3.4.15 In this section, we have shown that the Euclid statement is not a definite property of all real numbers. Neither 0, negative $\mathbb{R}_0$ numbers, nor a broad swath of positive $\mathbb{R}_{\aleph}^{\mathcal{X}}$ and $\widehat{\mathbb{R}}$ numbers have the property outlined by the Euclid statement and yet it is common to define the reals as $\mathbb{R} \equiv (-\infty, \infty)$. Indeed, even though we have used the name "Euclid statement" to refer to the definition based on natural numbers, Euclid did not explicitly require a natural multiplier, and neither did Archimedes nor Eudoxus. When we do not restrict the multiplier to $\mathbb{N}$, as we do not in the general statement of the Archimedes property, then every non-zero real number obeys the statement. According to Definition 3.4.5, all cuts in the real number line have the Archimedes property of real numbers. For this reason, the general statement of the Archimedes property which totally conforms to the quoted text from Euclid's Elements is superior to what we have called, purely for convenience, the Euclid statement.

Theorem 3.4.16 *All $\pm\mathbb{R}_{\aleph}^{\mathcal{X}}$ numbers obey the general Archimedes statement.*

Proof. The general Archimedes statement allows the multiplication of x by any real number z. Suppose $x < y$ and $x = \pm(\aleph_\mathcal{X} + b)$. By the arithmetic of $\mathbb{R}_\aleph^\mathcal{X}$ (Definition 2.3.27), we have

$$z(\pm\aleph_\mathcal{X} \pm b) = \pm\aleph_{(z\mathcal{X})} \pm zb \quad .$$

The Euclid statement requires $x < y$ and $nx > y$ but the general statement does not restrict to multiplication by naturals. Therefore, for $y \in \mathbb{R}_\aleph^{\mathcal{X}'}$ with $\mathcal{X} < \mathcal{X}'$ there exists a z such that $zx \in \mathbb{R}_\aleph^{\mathcal{X}''}$ with $\mathcal{X}'' > \mathcal{X}'$. It follows that

$$zx > y \quad .$$

Theorem 3.4.17 *All non-zero* $\mathbb{R}_0$, $\mathbb{R}_\aleph^\mathcal{X}$, *and* $\widehat{\mathbb{R}}$ *numbers obey the general Archimedes statement.*

Proof. Proof follows from the arithmetic operations of $\mathbb{R}_0$, $\mathbb{R}_\aleph^\mathcal{X}$, and $\widehat{\mathbb{R}}$.

Remark 3.4.18 In Section 4, we will show that there exist positive real numbers beyond those in $\mathbb{R}_0$, $\mathbb{R}_\aleph^\mathcal{X}$, and $\widehat{\mathbb{R}}$. (Note that this statement is semi-redundant because $\mathbb{R}_0 \subset \mathbb{R}_\aleph^0$ and $\widehat{\mathbb{R}} \subset \mathbb{R}_\aleph^1$.)

Conjecture 3.4.19 All non-zero real numbers obey the general statement of the Archimedes property of real numbers: for every real number, there exists a greater real number.

Remark 3.4.20 Assume that all real numbers obey the Archimedes property of real numbers. Any claim that the Euclid statement is (a) the golden standard of the Archimedes property and that (b) the multiplier is restricted to $\mathbb{N}$, necessarily entails that negative numbers are not real numbers. When a caveat about natural numbers is added to Euclid's words, we must also add a caveat about the signage of "magnitudes." Since it is very well possible, however, that Euclid's use of the word "multiplier" did, in fact, refer to an integer multiple $n \in \mathbb{N}$, we cannot definitely claim that he intended to allow any real multiplier $z \in \mathbb{R}$. If the multiplier seen in Euclid's words is indeed $n \in \mathbb{N}$, and only $n \in \mathbb{N}$, then it must be acknowledged that negative real numbers do not have the property he described. Therefore, Euclid's statement cannot be the golden standard of an ancient property imbued to all real numbers. When we add a caveat about the signage of magnitudes, we depart from anything that was written in antiquity and we are similarly well motivated to add a caveat about the neighborhood of the magnitudes. If one claims that it is obvious from the context that Euclid was only describing positive numbers, then it is equally obvious from the context that he was only describing numbers in the neighborhood of the origin. While the term "magnitudes" does imply strict positivity in the modern sense, Euclid is not said to have given the Archimedes property of magnitudes. Since Euclid is said to have given the Archimedes property of real numbers, it should be understood from the context that the word magnitudes is relied upon because the term “real numbers” did not exist in the language of the authors of that day. When magnitudes is taken as a synonym for real numbers, positivity is not strictly required because real numbers are not strictly positive.

Aside 3.4.21 The fractional distance framework of analysis developed in the present paper grew out of an earlier research inquiry in mathematical physics [2]. The MCM will be discussed in Section 6 but because the Archimedes property has a particularly keen bearing on a general point of physics, it is in order to make a brief aside in the present section. One of the main mysteries in quantum theory is the origin of the opposite phases of variation of the components of a fermionic spinor while the components of bosonic vectors vary in phase. One might examine the case in which the opposite phases of variation in spinors are derived from the reversal of the behavior under the Archimedes property between $\mathbb{R}_0$ and $\widehat{\mathbb{R}}$.

§4 The Topology of Real Numbers

Having demonstrated the existence of very many real numbers which have not been previously analyzed to any great degree, in this section we will apply the tools of elementary real mathematical analysis to develop the properties of these numbers. Toward the end of this section, we will demonstrate certain paradoxes and then propose, heuristically, things we might do to resolve them.

Theorem 4.1 $\mathbb{R}_\aleph^\mathcal{X} \subset \mathbb{R}_\infty$ *is an open set.*

Proof. $\mathbb{R}_\aleph^\mathcal{X}$ is open if and only if there is a δ-neighborhood of each of its elements such that every element of that neighborhood is also an element of $\mathbb{R}_\aleph^\mathcal{X}$. Elements of $\mathbb{R}_\aleph^\mathcal{X}$ have the form

$$x = \aleph_\mathcal{X} + b \quad , \qquad \text{where} \qquad b \in \mathbb{R}_\aleph^0 \not\subset \mathbb{R}_\infty \quad .$$

This theorem is proven with a δ-neighborhood of unit radius around an arbitrary $x \in \mathbb{R}_\aleph^\mathcal{X}$. We have

$$(x-\delta, x+\delta) = (\aleph_\mathcal{X} + b - \delta, \aleph_\mathcal{X} + b + \delta) = (\aleph_\mathcal{X} + b^-, \aleph_\mathcal{X} + b^+) \quad .$$

$b^\pm$ are such that

$$\forall b \in \mathbb{R}^0_\aleph \quad \exists \delta \in \mathbb{R}_0 \quad \text{s.t.} \quad b \pm \delta = b^\pm \in \mathbb{R}^0_\aleph \quad ,$$

so $\mathbb{R}^{\mathcal{X}}_\aleph$ is an open set because

$$(\aleph_{\mathcal{X}} + b^-, \aleph_{\mathcal{X}} + b^+) \subset \mathbb{R}^{\mathcal{X}}_\aleph \quad .$$

Theorem 4.2 *Given two neighborhoods $\mathbb{R}^{\mathcal{X}_1}_\aleph$ and $\mathbb{R}^{\mathcal{X}_2}_\aleph$ with $\mathcal{X}_1 < \mathcal{X}_2$, there exists another neighborhood $\mathbb{R}^{\mathcal{X}'}_\aleph$ such that $\mathcal{X}_1 < \mathcal{X}' < \mathcal{X}_2$.*

Proof. Consider the interval

$$[\aleph_{\mathcal{X}_1}, \aleph_{\mathcal{X}_2}] \subset \mathbb{R} \quad .$$

By Definition 2.3.8, the number at the center of this interval is

$$\frac{\aleph_{\mathcal{X}_2} + \aleph_{\mathcal{X}_1}}{2} = \aleph_{\left(\frac{\mathcal{X}_2+\mathcal{X}_1}{2}\right)} \quad .$$

We have

$$\mathcal{X}_1 < \left(\frac{\mathcal{X}_2 + \mathcal{X}_1}{2}\right) < \mathcal{X}_2 \quad ,$$

so let $\mathcal{X}' = (\mathcal{X}_2 + \mathcal{X}_1)/2$. Any number $x \in X$ of the form

$$x = \aleph_{\mathcal{X}'} + b \quad , \qquad \text{for} \qquad b \in \mathbb{R}^0_\aleph \quad ,$$

will be such that

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X}' \quad .$$

This proves the theorem.

Corollary 4.3 $\mathbb{R}^{\mathcal{X}_1}_\aleph \cup \mathbb{R}^{\mathcal{X}_2}_\aleph$ *is a disconnected set for any $\mathcal{X}_1 \neq \mathcal{X}_2$.*

Proof. A set is disconnected if it is the union of two disjoint, non-empty open sets. By Theorem 4.1, $\mathbb{R}^{\mathcal{X}_1}_\aleph$ and $\mathbb{R}^{\mathcal{X}_2}_\aleph$ are open, and it is obvious that they are non-empty. It follows from Theorem 4.2 that they are disjoint, *i.e.*:

$$\overline{\mathbb{R}}^{\mathcal{X}_1}_\aleph \cap \overline{\mathbb{R}}^{\mathcal{X}_2}_\aleph \equiv \emptyset \quad ,$$

where the bar denotes closure.

Main Theorem 4.4 *Any $\aleph_{\mathcal{X}}$ is an upper bound of $\mathbb{R}_0$.*

Proof. An upper bound of a set is greater than or equal to every element of that set. Given $\mathcal{X} > 0$, suppose

$$x \in \mathbb{R}_0 \quad , \quad x \in X \qquad \text{and} \qquad \aleph_{\mathcal{X}} \in Y \quad ,$$

so that

$$\mathcal{D}_{\mathbf{AB}}(AX) = 0 \quad , \qquad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AY) = \mathcal{X} \quad .$$

By the ordering of $\mathbb{R}$ (Definition 2.1.13), $\aleph_{\mathcal{X}}$ is an upper bound of $\mathbb{R}_0$

Corollary 4.5 $\mathbb{N}$ *is bounded from above.*

Proof. If $n \in \mathbb{N}$, then $n \in \mathbb{R}_0$. By Main Theorem 4.4, all $x \in \mathbb{R}_0$ are bounded from above. Therefore, $\mathbb{N}$ is bounded from above.

Remark 4.6 An axiom is a proposition regarded as self-evidently true without proof. Proposition 4.7 is central to Dedekind's 1872 characterization of real numbers with respect to ordered fields [4]. Dedekind likely formulated the following proposition without first considering the properties of real numbers in the neighborhood of infinity.

Proposition 4.7 Every non-empty set of real numbers which is bounded from above has a least upper bound (supremum.)

Refutation. Corollary 4.5 gives an upper bound for $\mathbb{N}$ so this proposition may be refuted by showing that $\mathbb{N}$ does not have a least upper bound. To invoke a contradiction, suppose $x \in \mathbb{R}$ is a least upper bound of $\mathbb{N} \subset \mathbb{R}$. For any $n \in \mathbb{N}$, we have

$$n \leq x \qquad \Longrightarrow \qquad n - 1 \leq x - 1 \quad .$$

Every $n' \in \mathbb{N}$ can be written as $n - 1$ so we obtain

$$n' \leq x - 1 \quad ,$$

which contradicts the supposition that x is a least upper bound of $\mathbb{N}$.

Definition 4.8 $\widehat{\mathbb{R}}_{\aleph}$ is a subset of all real numbers

$$\widehat{\mathbb{R}}_{\aleph} = \left\{ \pm(\widehat{\infty} - b) \mid b \in \mathbb{R}_{\aleph}^{0} \ , \ b > 0 \right\} \quad .$$

This definition generalizes $\widehat{\mathbb{R}}$ to the case of $b \in \mathbb{R}_{\aleph}^{0}$.

Definition 4.9 $\mathbb{R}'$ is the union

$$\mathbb{R}' \equiv \bigcup_{0<\mathcal{X}<1} \pm\mathbb{R}_\aleph^\mathcal{X} \cup \pm\mathbb{R}_\aleph^0 \cup \widehat{\mathbb{R}}_\aleph \quad .$$

Definition 4.10 The positive subsets of $\mathbb{R}'$ are

$$\begin{aligned} \mathbb{R}_{\aleph+}^0 &\subset \pm\mathbb{R}_\aleph^0 \\ \mathbb{R}_\aleph^\mathcal{X} &\subset \pm\mathbb{R}_\aleph^\mathcal{X} \\ \widehat{\mathbb{R}}_\aleph^+ &\subset \widehat{\mathbb{R}}_\aleph \\ \mathbb{R}'_+ &\subset \mathbb{R}' \quad . \end{aligned}$$

$\mathbb{R}_{\aleph+}^0$ is specified because $\{0\} \in \mathbb{R}_\aleph^0$ is non-positive.

Main Theorem 4.11 *There exist more real numbers than are in* $\mathbb{R}'$*.*

Proof. By definition, the interval $(-\infty, \infty) \equiv \mathbb{R}$ is a connected set. To prove the present theorem, it will suffice to show that $\mathbb{R}'_+$ is disconnected. To frame a classical statement of discontinuity, let

$$U = \mathbb{R}_{\aleph+}^0 \quad , \qquad \text{and} \qquad V = \mathbb{R}'_+ \setminus \mathbb{R}_{\aleph+}^0 \quad .$$

If the open set $\mathbb{R}'_+$ is disconnected then

$$U \cup V = \mathbb{R}'_+ \quad , \qquad \text{and} \qquad U \cap V = \emptyset \quad .$$

The former condition is satisfied trivially. The latter condition is proven by contradiction. Assume

$$U \cap V = x \quad , \qquad \text{where} \qquad x \in (0, \infty) \subset \mathbb{R} \quad .$$

It follows that x is the algebraic representation of some point $X \in \mathbf{AB}$, and Theorem 2.1.27 proves that if $x \in Y$ then $X = Y$. Then

$$x \in U \quad \implies \quad \mathcal{D}_{\mathbf{AB}}(AX) = 0 \quad ,$$

and

$$x \in V \quad \implies \quad \mathcal{D}_{\mathbf{AB}}(AX) > 0 \quad .$$

This contradicts the one-to-one property of $\mathcal{D}_{\mathbf{AB}}(AX)$ (Theorem 2.1.14) and, therefore, $U \cap V = \emptyset$.

Alternatively, proof follows from Corollary 4.3. ∎

Theorem 4.12 *It is not possible to construct a Cantor set on* $\mathbf{AB}$*.*

Proof. Cantor sets contain no intervals but every $X \in \mathbf{AB}$ has an algebraic representation as an interval. Therefore, if any of the points of $\mathbf{AB}$ remain after some construction by iterative deletions, that construction will necessarily contain algebraic intervals of real numbers.

Definition 4.13 F_∞ is a Cantor-like set on $\mathbf{AB}$ constructed as follows. Begin a recursive construction

$$\begin{aligned}\mathbf{AB} \equiv F_0 &= [0,\infty] \\ F_1 &= F_0 \setminus \left\{\mathbb{R}^0_{\aleph+} \cup \widehat{\mathbb{R}}^+_\aleph\right\} \\ &= \{0,\infty\} \bigcup_{0<\mathcal{X}<1} \mathbb{R}^{\mathcal{X}}_\aleph \quad .\end{aligned}$$

Choose $0 < \mathcal{X}_1 < 1$ and let

$$\begin{aligned}F_2 &= F_1 \setminus \mathbb{R}^{\mathcal{X}_1}_\aleph \\ &= \{0,\infty\} \bigcup_{0<\mathcal{X}<\mathcal{X}_1} \mathbb{R}^{\mathcal{X}}_\aleph \bigcup_{\mathcal{X}_1<\mathcal{X}'<1} \mathbb{R}^{\mathcal{X}'}_\aleph \quad .\end{aligned}$$

Choose $0 < \mathcal{X}_{01} < \mathcal{X}_1$ and $\mathcal{X}_1 < \mathcal{X}_{11} < 1$ so that

$$\begin{aligned}F_3 &= F_2 \setminus \left\{\mathbb{R}^{\mathcal{X}_{01}}_\aleph \cup \mathbb{R}^{\mathcal{X}_{11}}_\aleph\right\} \\ &= \{0,\infty\} \bigcup_{0<\mathcal{X}<\mathcal{X}_{01}} \mathbb{R}^{\mathcal{X}}_\aleph \bigcup_{\mathcal{X}_{01}<\mathcal{X}'<\mathcal{X}_1} \mathbb{R}^{\mathcal{X}'}_\aleph \bigcup_{\mathcal{X}_1<\mathcal{X}''<\mathcal{X}_{11}} \mathbb{R}^{\mathcal{X}''}_\aleph \bigcup_{\mathcal{X}_{11}<\mathcal{X}'''<1} \mathbb{R}^{\mathcal{X}'''}_\aleph \quad .\end{aligned}$$

By recursion, we construct a Cantor-like set F_∞.

Definition 4.14 The complement of $\mathbb{R}_0$ in $\pm\mathbb{R}^0_\aleph$ is $\mathbb{R}^C_0$:

$$\mathbb{R}_0 \cup \mathbb{R}^C_0 = \pm\mathbb{R}^0_\aleph \quad .$$

Theorem 4.15 *If $x \in F_\infty$ and $x \neq \infty$, then $x \in \mathbb{R}$.*

Proof. Observe that

$$x \in F_\infty \setminus \infty \subset [0,\infty) \subset \mathbb{R} \quad .$$

Corollary 4.16 *$\mathbb{R}$ is the union of $\mathbb{R}'$ and $\pm F_\infty$.*

Proof. Proof follows from Definitions 4.9 and 4.13.

Remark 4.17 Recall the distinction between $\mathbb{R}_0$ and $\mathbb{R}^0_\aleph$: there might be numbers larger than the largest natural number but less than any number in

$\mathbb{R}_{\aleph}^{\mathcal{X}}$ when $\mathcal{X} > 0$. $\mathbb{R}_{\aleph}^{0}$ would contain these numbers while $\mathbb{R}_0$ cannot. What are these numbers $x \in \mathbb{R}_0^C$? We have not described them, but they appear as part of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ in Definition 2.3.12:

$$\mathbb{R}_{\aleph}^{\mathcal{X}} = \{\aleph_{\mathcal{X}} \pm b \mid \mathcal{X} > 0 \ , \ b \in \mathbb{R}_{\aleph}^{0}\} \ \ .$$

Since we do not have a a definition for $\mathbb{R}_0^C$, it might be better to restrict $\mathbb{R}_{\aleph}^{\mathcal{X}}$ as

$$\mathbb{R}_{\aleph}^{\mathcal{X}} = \{\aleph_{\mathcal{X}} \pm b \mid \mathcal{X} > 0 \ , \ b \in \mathbb{R}_0\} \ \ .$$

letting numbers of the form

$$x' = \aleph_{\mathcal{X}} + b \ \ , \qquad \text{with} \qquad b \in \mathbb{R}_0^C \ \ ,$$

be reassigned to some other subsets of $\mathbb{R}$ which are neighborhoods of the elements of F_∞. Although we have shown in Theorem 4.2 that we cannot use the $\mathbb{R}_{\aleph}^{\mathcal{X}}$ notation to define two adjacent neighborhoods $\mathbb{R}_{\aleph}^{\mathcal{X}_1}$ and $\mathbb{R}_{\aleph}^{\mathcal{X}_2}$ not separated by a similar neighborhood $\mathbb{R}_{\aleph}^{\mathcal{X}'}$, this is merely an artifact of the insufficiency of the infinite decimal representation of $\mathcal{X}$. By the connected property of $\mathbb{R}$, we know that there every $\mathbb{R}_{\aleph}^{\mathcal{X}}$ neighborhood has a least greater $\mathbb{R}_{\aleph}^{\mathcal{X}}$ neighborhood and greatest lesser $\mathbb{R}_{\aleph}^{\mathcal{X}}$ neighborhood. We will call these "adjacent" $\mathbb{R}_{\aleph}^{\mathcal{X}}$ neighborhoods. While we will not do so presently, we could easily enumerate these adjacent neighborhoods with natural numbers such that there is a first one, a second, *etc.* Although these adjacent neighborhoods are not separated by other $\mathbb{R}_{\aleph}^{\mathcal{X}}$ neighborhoods, we have defined F_∞ such that its elements do separate adjacent $\mathbb{R}_{\aleph}^{\mathcal{X}}$ neighborhoods. This gives a good reason to include numbers like x' in some standardized δ-neighborhood of the elements of F_∞ rather than in the standardized δ-neighborhood of $\aleph_{\mathcal{X}}$ which we call $\mathbb{R}_{\aleph}^{\mathcal{X}}$.

Paradox 4.18 Every set $\mathbb{R}_{\aleph}^{\mathcal{X}}$ has a geometric representation as a point $X \in$ **AB** such that

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \ \ .$$

However, the geometric representation of a single number $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ is the same point X, presumably, that is the geometric representation of the entire neighborhood $\mathbb{R}_{\aleph}^{\mathcal{X}}$. When constructing F_∞ (Definition 4.13), the first iterative deletion F_1 requires that we delete positive $\mathbb{R}_{\aleph}^{0}$ numbers ($\mathbb{R}_{\aleph+}^{0}$.) If we deleted all $\mathbb{R}_{\aleph}^{0}$ numbers then that would be a geometric deletion of the left endpoint $A \in$ **AB** but $0 \in [0, \widehat{\infty}]$ is not deleted from F_1 because 0 is non-positive. By Theorem 2.4.2, if A remains, then A is an interval. However, any connected δ-neighborhood of 0 would contain positive $\mathbb{R}_0$ numbers which have already been deleted. This is a stark paradox.

Paradox Resolution 4.19 Paradox 4.18 hinges on the geometric representation of algebraic numbers. To this point, we have assumed that if the algebraic representation of a geometric point consists of numbers then the algebraic representation of numbers must be points. However, we have now shown that an assumption of commutativity in such reasoning induces an unwanted paradox. To resolve it, we must cease to assume that we may switch between the geometric and algebraic analytical frameworks in an *ad hoc* way. We have required in the present analysis that points may be converted to numbers, but nothing in the present analysis has strictly required that numbers may be converted to points. Indeed, we get a hint of the one-way nature of the relationship between geometry and algebra when geometric infinity is absolutely large but algebraic infinity can be made arbitrarily small though conformal transformations.

Consider the point density of points. Usually it is assumed that the point density of points is unity meaning that every point contains exactly one point. If we increase the point density of points from unity to infinity, namely that we impose fractal self-similarity on points, then we gain a lot of freedom with respect what the geometric representation of arbitrary sets of algebraic numbers might be. As an example of the flavor of the idea, consider the argument that $\widehat{\mathbb{R}}_{\aleph}^{+}$ cannot contain distinct numbers because we have added only a single point $\{\widehat{\infty}\}$ to the positive branch of $\mathbb{R}$. This argument assumes that what have traditionally been called real numbers ($\mathbb{R}_0$) are spread out along some long extent of the real number line, and then we only add a single point at the end. In the present analysis, however, we have shown all $x \in \mathbb{R}_0$ are compactified down at the left endpoint of $\mathbf{AB}$ because the largest classically real number has zero fractional distance with respect to infinity. Therefore, the argument that $\widehat{\mathbb{R}}_{\aleph}^{+}$ can't contain unique numbers, if sound, would also imply that $\mathbb{R}_0$ can't contain many different real numbers. Obviously such an argument is not sound.

Indeed, there is an interesting and nearly paradoxical result in classical analysis which shows that the number of points in even the shortest line segment is exactly equal to the number of points in the largest conceivable N-dimensional volume. The density of points in any continuum in any dimension is infinity. By extending the infinite density even down to the resolution of single points, it becomes more intuitive that every continuum should have an equal number of points. The infinite point density of points should allow us to add unique identifiers to points $X \in AB$ such that we resolve Paradox 4.18 directly. Rather than saying that $A \in \mathbf{AB}$ remains at the F_1 step, we may avert the paradox by introducing notation such that $A_0 \in \mathbf{AB}$ remains. Here we would essentially conjure the notion of points which are infinitesimal with respect to other points, but the reader should carefully notice that this does not require any real numbers to be infinitesimal with respect to other real numbers. In the suggested notation, A_0 is the point of $A \in \mathbf{AB}$ which remains after $\mathbb{R}_0^{+}$ is deleted from $\mathbf{AB}$. We will not fully implement this idea for infinitesimal points in the present paper. To do so we would have to address Definition 1.4.1: A line segment is a line with two different points. We would have to carefully review all results reported here to determine whether or not $A_x A_y$ is a line segment or a point. It should be an infinitesimal line segment, but it is not immediately clear if such should be considered to be a pointlike object.

Aside 4.20 Before identifying non-trivial zeros of the Riemann ζ function off the critical line in earlier work [1], we first proved the existence of those zeros in Reference [5]. This aside pertains to the earlier analysis in Reference [5]. In that paper whose main result appears here in Section 5, we closely examined the polar point of the Riemann sphere which remains uncharted when a single coordinate chart is painted onto $\mathbb{S}^2$. For that analysis, we defined a series of increasingly infinitesimal discs around the polar point called "the hypercomplexly infinitesimal neighborhood" of the polar point. This neighborhood has a direct analogue on the present work and would certainly pertain to infinitesimal line segments as the radii of the infinitesimal discs. When $\mathbb{R}$ is mapped to a great circle of the Riemann sphere, the polar ray through any zenith in $[0, \pi)$ points to some $\mathbb{R}_0$ number. Therefore, the infinite series of increasingly small concentric discs in the hypercomplexly infinitesimal neighborhood of the polar point is in direct, one-to-one correspondence with the infinite series of increasingly large $\mathbb{R}_\aleph^\mathcal{X}$ neighborhoods described in the present analysis. Furthermore, when studying the hypercomplexly infinitesimal neighborhood of the Riemann sphere's polar point, it was required to give different properties to consecutive discs on approach to the polar point at the center of an infinite number of nested discs. When the discs were labeled with sequential natural numbers, two classes of properties emerged: one for even numbered discs and one for odd. In the present analysis, this is reflected in the alternating neighborhoods of $\aleph_\mathcal{X}$ and $\mathcal{F}_n$. In the earlier analysis [5], we studied approach to the polar point on $\mathbb{S}^2$ and here we use $\mathbb{R}_\aleph^\mathcal{X}$ and $\mathcal{F}_n$ to study the linear approach to infinity at the end of the real number line.

The final note in this aside pertains to the definition of infinitesimal points such as the $A_0 \in A \in AB$ proposed in Paradox Resolution 4.19. In reference [5], we introduced labels called "levels of aleph" such that the charted part of the Riemann sphere was on the first level, the first region inside the hypercomplexly infinitesimal neighborhood was on the second, the doubly infinitesimal disc at the center of the first infinitesimal disc was on the third level of aleph, *etc.* We proposed as a workaround for dealing with infinity (in the context of Reference [5]) to relabel the levels of aleph so that the first infinitesimal disc could be considered finite and the doubly infinitesimal disc at its center could be considered ordinarily infinitesimal. The procedure was described as "going to a higher level of aleph" and by allowing points to be infinitesimal with respect to other points, as in Paradox Resolution 4.19, we are essentially putting the geometric representation of AB "on a higher level of aleph," call it the n^{th} level. The algebraic representation, then, is "on a lower level," call it the m^{th} level where $m = n - 1$. This allows us to have "infinitesimal points" without requiring infinitesimal real numbers. Since there is no such thing as an infinitesimal real number, such a scheme would be in very good order.

Definition 4.21 Every connected interval in F_∞ has a number at its center (Definition 2.2.1.) Call these numbers $\mathcal{F}_n$ for $n \in \mathbb{N}$ such that

$$n > m \quad \Longrightarrow \quad \mathcal{F}_n > \mathcal{F}_m \quad .$$

Paradox 4.22 By the connected property of $\mathbb{R}$, every $\mathcal{F}_n$ is in the algebraic representation of some interior point $X \in \mathbf{AB}$. All such X have the property

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \quad , \qquad \text{where} \qquad \mathcal{X} \in (0,1) \quad .$$

This is a paradox because it implies $\mathcal{F}_n \in \mathbb{R}_\aleph^\mathcal{X}$ while we have constructed F_∞ by removing every $\mathbb{R}_\aleph^\mathcal{X}$ from $\mathbb{R}$.

Remark 4.23 As a matter of practicality, Paradox 4.22 demonstrates an insufficiency of the infinite decimal representation of $\mathcal{X}$.

Definition 4.24 Every number between two adjacent $\mathbb{R}_\aleph^\mathcal{X}$ neighborhoods is contained in a neighborhood of $\mathcal{F}_n$ called $\mathbb{R}_\mathcal{F}^n$.

Theorem 4.25 *$\mathbb{R}_\mathcal{F}^n$ is a closed set.*

Proof. Let $z \in \mathbb{R}_\aleph^\mathcal{X}$ for some $\mathcal{X} > 0$. It follows from Definition 4.24 that every number in the interval $(0,z)$ is either $x \in \mathbb{R}_\aleph^\mathcal{X}$ for some $\mathcal{X}$ or $x \in \mathbb{R}_\mathcal{F}^n$ for some $n \in \mathbb{N}$. Since all $\mathbb{R}_\aleph^\mathcal{X}$ are open sets, it follows from the connected property of $\mathbb{R}$ that all $\mathbb{R}_\mathcal{F}^n$ are closed sets.

Theorem 4.26 *The infimum of $\mathbb{R}_\mathcal{F}^1$ is the supremum of $\mathbb{R}_\aleph^0$.*

Proof. Let $\mathbb{R}_{\mathcal{F}-}^1$ be the set of $\mathbb{R}_\mathcal{F}^1$ number less than or equal to $\mathcal{F}_1$. By the connected property of $\mathbb{R}$, we have

$$[0,\mathcal{F}_1] = \mathbb{R}_\aleph^0 \cup \mathbb{R}_{\mathcal{F}-}^1 \quad .$$

We have proven in Theorem 4.25 that $\mathbb{R}_\mathcal{F}^1$ is a closed set, and it follows that $\mathbb{R}_\mathcal{F}^1$ has an infimum. Therefore,

$$\sup \mathbb{R}_\aleph^0 = \inf \mathbb{R}_\mathcal{F}^1 \quad .$$

Remark 4.27 Theorem 4.26 does not require a supremum for $\mathbb{R}_0$ or $\mathbb{N}$ because these numbers are separated from $\inf \mathbb{R}_\mathcal{F}^1$ by $\mathbb{R}_0^C$.

Theorem 4.28 *For any $x \in \mathbb{R}_\aleph^0$ such that $x \in X$, and for $X \in AB$ such that $AB \equiv [0, \mathcal{F}_1]$, we have*

$$\mathcal{D}_{AB}(AX) = 0 \quad .$$

Proof. $\mathcal{D}_{AB}(AX)$ is such that

$$\mathcal{D}_{AB}(AX) = \mathcal{D}^\dagger_{AB}(AX) = \frac{x}{\mathcal{F}_1} \quad .$$

For proof by contradiction, suppose $z > 0$ and that

$$\frac{x}{\mathcal{F}_1} = z \quad .$$

Since $\|x\| < \|\mathcal{F}_1\|$, it follows that $z \in \mathbb{R}_0$. All such z have a multiplicative inverse so

$$\frac{x}{z\mathcal{F}_1} = 1 \qquad \Longleftrightarrow \qquad z^{-1}x = \mathcal{F}_1 \quad .$$

The condition $x \in \mathbb{R}_\aleph^0$ requires

$$x < \aleph_\mathcal{X} \quad \forall \mathcal{X} > 0 \quad ,$$

so for any non-zero $z \in \mathbb{R}_0$ we have

$$z^{-1}x = z^{-1}b < \aleph_{(z^{-1}\mathcal{X})} \quad \forall \mathcal{X} > 0 \qquad \Longrightarrow \qquad z^{-1}x \in \mathbb{R}_\aleph^0 \quad .$$

This delivers a contradiction because $\mathcal{F}_1$ is larger than any $\mathbb{R}_\aleph^0$ number and cannot be equal to $z^{-1}x$.

Corollary 4.29 *For any $x \in \mathbb{R}_\aleph^0$ and $AB \equiv [0, \mathcal{F}_1]$, it must be that $x \in A$.*

Proof. Proof follows from Theorem 4.28.

Remark 4.30 At this point, we have some guidance regarding the numbers in $\mathbb{R}_0^C$. To begin to define them, we would use fractional distance functions to define numbers that are $(100 \times \mathcal{X})\%$ of the way to $\mathcal{F}_1$. Then we would define a Cantor-like set on $AB = [0, \mathcal{F}_1]$ and show that $\mathbb{R}_\aleph^0$ only covers 0% of the distance from zero to the least element of that Cantor-like set. This would go on and on forever as we develop an inverse Zeno's paradox of stalled growth prohibiting $x \in \mathbb{R}_\aleph^0$ from ever achieving some non-zero fractional distance by leaving the point $A \in AB$.

Paradox 4.31 If we have constructed a Cantor-like set F_∞ by removing all $\mathbb{R}_\aleph^\mathcal{X}$ from **AB**, then it would follow that almost all of the distance along **AB** is covered by $\mathbb{R}_\aleph^\mathcal{X}$. In Theorem 4.28, however, we have shown that the $\mathbb{R}_\aleph^\mathcal{X}$ are like a Cantor dust having zero fractional distance. If these neighborhoods have very little fractional distance along **AB**, then the complementary neighborhoods $\mathbb{R}^n_{\mathcal{F}}$ must have almost all of the fractional distance along **AB**. Since all the numbers in $\mathbb{R}^n_{\mathcal{F}}$ are in the algebraic representation of the points which continue to exist after the infinitieth deletion leading to F_∞, we would expect that these $\mathbb{R}^n_{\mathcal{F}}$ neighborhoods comprise very little of the total fractional distance along **AB**. However, Theorem 4.28 shows that these neighborhoods have almost all of the total fractional distance.

Paradox Resolution 4.32 Paradox 4.31 can be resolved in the geometric picture. There are an uncountably infinite number of points in **AB**, and each $\mathbb{R}^{\mathcal{X}}_{\aleph}$ is the point for which the fractional distance is equal to $\mathcal{X}$. We remove a finite number of these points in each step F_n, and we repeat this only countably infinitely many times to construct F_∞ because $n \in \mathbb{N}$. Therefore, after removing countably infinitely many points from **AB**, uncountably infinitely many points will yet remain.

Remark 4.33 In Remark 4.17, we proposed to restrict $\mathbb{R}^{\mathcal{X}}_{\aleph}$ to numbers of the form

$$x = \aleph_{\mathcal{X}} + b \quad , \qquad \text{where} \qquad b \in \mathbb{R}_0 \quad ,$$

rather than the form

$$x = \aleph_{\mathcal{X}} + b \quad , \qquad \text{where} \qquad b \in \mathbb{R}^0_{\aleph} \quad ,$$

If we construct an alternative version of F_∞ through iterative deletions of restricted version of $\mathbb{R}^{\mathcal{X}}_{\aleph}$, then the numbers removed from the restricted neighborhood, numbers of the form

$$x = \aleph_{\mathcal{X}} + b \quad , \qquad \text{where} \qquad b \in \mathbb{R}^C_0 \quad ,$$

will not be deleted from the parent interval **AB** at any constructive step leading to F_∞. Instead, these numbers will be included in the $\mathbb{R}^n_{\mathcal{F}}$.

Definition 4.34 The restricted set of numbers that are $(100 \times \mathcal{X})\%$ of the way down the real number line is

$$\mathbb{R}^{\mathcal{X}}_0 = \{\aleph_{\mathcal{X}} \pm b \mid b \in \mathbb{R}_0\} \quad .$$

Definition 4.35 F'_∞ is a Cantor-like set on **AB** constructed as follows. Begin a recursive construction

$$\begin{aligned} \mathbf{AB} \equiv F'_0 &= [0, \infty] \\ F'_1 &= F'_0 \setminus \left\{ \mathbb{R}^0_{0+} \cup \widehat{\mathbb{R}}^+_{\aleph} \right\} \end{aligned}$$

$$= \{0, \infty\} \bigcup_{0<\mathcal{X}<1} \mathbb{R}^{\mathcal{X}}_{\aleph} \quad .$$

Choose $0 < \mathcal{X}_1 < 1$ and let

$$\begin{aligned} F_2' &= F_1 \setminus \mathbb{R}^{\mathcal{X}_1}_0 \\ &= \{0, \infty\} \bigcup_{0<\mathcal{X}<\mathcal{X}_1} \mathbb{R}^{\mathcal{X}}_{\aleph} \bigcup_{\mathcal{X}_1<\mathcal{X}'<1} \mathbb{R}^{\mathcal{X}'}_{\aleph} \cup \mathbb{R}^{\mathcal{X}_1}_{\aleph} \setminus \mathbb{R}^{\mathcal{X}_1}_0 \quad . \end{aligned}$$

Choose $0 < \mathcal{X}_{01} < \mathcal{X}_1$ and $\mathcal{X}_1 < \mathcal{X}_{11} < 1$ so that by recursion, we construct a Cantor-like set F'_∞.

Paradox 4.36 In the construction of both of F_∞ and F'_∞, we have used the algebraic representation to delete open intervals from the parent interval $\mathbf{AB} = [0, \widehat{\infty}]$. By using $\mathbb{R}^{\mathcal{X}}_0$ in the construction of F'_∞ as opposed to $\mathbb{R}^{\mathcal{X}}_{\aleph}$ in the construction of F_∞, we require that the length of the interval representation of each $\mathbb{R}^n_{\mathcal{F}}$ increase with respect to what it was when we deleted $\mathbb{R}^{\mathcal{X}}_{\aleph}$. However, in the geometric picture, iterative deletion of either $\mathbb{R}^{\mathcal{X}}_0$ or $\mathbb{R}^{\mathcal{X}}_{\aleph}$ should be the deletion of a point. In both of F_∞ and F'_∞, we delete the same number of points at each of the same number of steps, and we should arrive at the same final Cantor-like set. However, we know that the algebraic representations of the elements of F'_∞ must be intervals of greater length than algebraic representations of the elements of F_∞. Therefore, we have a paradox showing that two objects are both equal and unequal.

Paradox Resolution 4.37 Paradox 4.36 may be resolved by the introduction of infinitesimal points as suggested in Paradox Resolution 4.19.

Remark 4.38 Paradox Resolution 4.37 is very broad. After we develop one final paradox, we will propose a resolution which more concisely solves Paradox 4.36 than what is proposed in Paradox Resolution 4.37.

Paradox 4.39 In constructing both of F_∞ and F'_∞, we have followed the prescription for constructing a Cantor set and yet we do not obtain one. If we switched to the conformal chart so that $[0, \widehat{\infty}] \to [0, \pi/2]$, and then followed the same constructive steps by the deletion of open sets, we would definitely produce a Cantor set. It is paradoxical that we have destroyed the Cantor set through our choice of chart on AB.

Paradox Resolution 4.40 Paradox 4.39 can be avoided if we endow $\aleph_{\mathcal{X}}$ with a constrained property of additive absorption:

$$b \in \mathbb{R}^0_{\aleph} \quad \Longrightarrow \quad \aleph_{\mathcal{X}} + b = \aleph_{\mathcal{X}} \quad .$$

Given this property, there is only one number in each $\mathbb{R}^{\mathcal{X}}_{\aleph}$. The prescriptions for F_∞ and F'_∞ cease to be the recipe for a Cantor set because $\mathbb{R}^{\mathcal{X}}_{\aleph}$ become closed sets and the $\mathbb{R}^{n}_{\mathcal{F}}$ become open. Indeed, since we have proposed a hat in Section 3.1 to suppress absorption, we might introduce

$$\widehat{\mathbb{R}}^{\mathcal{X}}_{\aleph} = \{\widehat{\aleph}_{\mathcal{X}} \pm b \mid \mathcal{X} > 0 \ , \ b \in \mathbb{R}^0_{\aleph}\} \quad .$$

Since the non-absorptivity of $\aleph_{\mathcal{X}}$ induces Paradox 4.39, we may invoke the non-contradiction property of Definition 3.1.11 to require the removal of the hat when constructing F_∞. Indeed, if we imbue $\aleph_{\mathcal{X}}$ with the constrained property of additive absorption, then it is only the endpoints of **AB** which have the curious property of having more than one number in their algebraic representations. We will not rigorously introduce $\widehat{\aleph}_{\mathcal{X}}$ here but such a number would clearly have many favorable properties such as resolving the limit issue raised in Remark 3.1.21.

Example 4.41 This examples shows that the additive absorptive property for $\aleph_{\mathcal{X}}$ is supported by the ordinary notions of trigonometry. Let ABC be a right triangle such that $\angle ABC = \pi/2$ and

$$\|AB\| \in \mathbb{R}^{\mathcal{X}_1}_{\aleph} \quad , \qquad \text{and} \qquad \|BC\| \in \mathbb{R}^{\mathcal{X}_2}_{\aleph} \quad .$$

Trigonometry requires

$$\angle ACB = \tan^{-1}\left(\frac{\|AB\|}{\|BC\|}\right) \quad .$$

By Definition 2.3.28, however, this fraction is undefined for

$$\|AB\| \neq c\|BC\| \quad .$$

By adding constrained additive absorption, there will always exist some $c = \mathcal{X}_1/\mathcal{X}_2$ such that

$$\aleph_{\mathcal{X}_1} = c\aleph_{\mathcal{X}_2} = \aleph_{(c\mathcal{X}_2)} \quad .$$

While it solves the problem of the undefined angle, additive absorptive is not the only solution and we can cannot require it as a property of $\aleph_{\mathcal{X}}$ solely on this example. Another workaround would be to revise the arithmetic operations (Definition 2.3.28) such that

$$\forall b_1, b_2 \in \mathbb{R}^0_{\aleph} \quad \exists x_1 = \aleph_{\mathcal{X}_1} + b_1 \quad \exists x_2 = \aleph_{\mathcal{X}_2} + b_2 \quad \text{s.t.} \quad \frac{x_1}{x_2} = \frac{\mathcal{X}_1}{\mathcal{X}_2} \quad .$$

As given in Definition 2.3.28, x_1/x_2 is undefined in all but a few special cases. If we added a definition for the quotient in the general case, that would contradict Theorem 2.3.29 but we could still make another workaround by taking away the multiplicative inverse of $x = \aleph_{\mathcal{X}} + b$ for non-zero b.

Remark 4.42 Following in the direction of Example 4.41, and using the same right triangle ABC, one could use the Pythagorean theorem to derive a result about the square root of $x \in \mathbb{R}^{\mathcal{X}}_{\aleph}$.

§5 The Riemann Hypothesis

In this section, we negate the Riemann hypothesis at the highest possible standard of mathematical rigor by showing non-trivial zeros of ζ in the neighborhood of infinity. First we will refute two common claims, one not so rigorous, that ζ cannot have a zero at any z such that $\text{Re}(z) > 1$. Then we will show that Riemann's functional continuation of the Dirichlet form of ζ does not absolutely converge on all of $\mathbb{C}$. After that, we will show that ζ has an infinite number of zeros in the neighborhood of infinity.

Proposition 5.1 The Riemann $\zeta(z)$ function cannot have zeros in the region $\text{Re}(z) > 1$ because all of the terms in the Euler product to which it absolutely converges in that region are non-zero.

Refutation. Consider the inverse Riemann ζ function

$$\frac{1}{\zeta(z)} = \prod_p \left(1 - p^{-z}\right) \quad .$$

Observe that $\zeta^{-1}(z)$ takes $z = 1$ as

$$\frac{1}{\zeta(1)} = \frac{1}{\widehat{\infty}} = 0 = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)... \quad .$$

None of the terms in the infinite product are equal to zero and yet the function $\zeta^{-1}(z)$ to which it is equal is equal to zero.

Proposition 5.2 If $\text{Re}(z) > 1$, then $\zeta(z) \neq 0$ because

$$\zeta(z)\prod_p(1 - p^{-z}) = 1 \quad .$$

Refutation. The argument in favor of this proposition goes as follows [6]. In the region $\text{Re}(z) > 1$, $\zeta(z)$ absolutely converges to the Euler product so

$$\zeta(z) = \prod_p \frac{1}{1 - p^{-z}} \qquad \Longrightarrow \qquad \zeta(z)\prod_p(1 - p^{-z}) = 1 \quad .$$

Every prime number is a natural number, so every term in $\Pi_p\,(1-p^{-z})$ is contained in $\Pi_n\,(1-n^{-z})$. It follows that the former product will converge if the latter does. It is a property of infinite products that $\Pi_n\,(1+a_n)$ converges if and only if $\Sigma_n\, a_n$ converges. Since $\Sigma_n\, n^{-z}$ does converge for $\mathrm{Re}(z)>1$, we know that $\Pi_n\,(1-n^{-z})$ absolutely converges. Therefore, $\Pi_p\,(1-p^{-z})$ absolutely converges and the condition that

$$\zeta(z)\prod_p(1-p^{-z})=1 \quad ,$$

guarantees that $\zeta(z)\neq 0$ for any z such that $\mathrm{Re}(z)>1$. If there was some z_0 such that $\zeta(z_0)$ was equal to zero, then the expression could not be equal to one. To show the failure of this argument, consider $z_0\in\widehat{\mathbb{R}}$ such that

$$z_0=\widehat{\infty}-b \qquad \Longrightarrow \qquad \mathrm{Re}(z_0)>1 \quad .$$

Then,

$$\prod_p(1-p^{-\widehat{\infty}+b})=\prod_p\left(1-\frac{p^b}{\widehat{\infty}}\right)=\lim_{x\to\widehat{\infty}}\left(1-\frac{p^b}{\widehat{\infty}}\right)^x \quad .$$

This is an indeterminate form because $\widehat{\infty}$ is defined as a limit (Definition 1.3.2). Therefore, the expression

$$\zeta(z)\prod_p(1-p^{-z})=1 \quad ,$$

cannot be used to rule out zeros of ζ for all z with $\mathrm{Re}(z)>1$. It might be true that there are no zeros in the region $\mathrm{Re}(z)>1$ but the matter cannot be proven by the reasoning given in Proposition 5.2.

Definition 5.3 A number is a complex number $z\in\mathbb{C}$ if and only if

$$z=x+iy \quad , \qquad \text{and} \qquad x,y\in\mathbb{R} \quad .$$

Theorem 5.4 *The Riemann ζ function is equal to one for any $z_0\in\mathbb{R}_{\aleph}^{\mathcal{X}}$ such that $0<\mathcal{X}<1$.*

Proof. Observe that the Dirichlet sum form of ζ

$$\zeta(z)=\sum_n\frac{1}{n^z} \quad ,$$

takes $z_0=\aleph_{\mathcal{X}}+b$ as

$$\zeta(\aleph_{\mathcal{X}}+b)=\sum_{n=1}\frac{1}{n^{\aleph_{\mathcal{X}}+b}}=\sum_{n=1}\frac{n^{-b}}{n^{\mathcal{X}\aleph_1}}=1+\sum_{n=2}\frac{n^{-b}}{\widehat{\infty}}=1 \quad .$$

Theorem 5.5 *The Riemann ζ function has non-trivial zeros with real parts in $-\mathbb{R}_{\aleph}^{\mathcal{X}}$ for $0 < \mathcal{X} < 1$.*

Proof. Riemann's functional form of ζ [7] is

$$\zeta(z) = 2(2\pi)^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad ,$$

and we will prove this theorem with the Euler definition of the Γ function [8]

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\widehat{\infty}} \frac{\left(1+\dfrac{1}{n}\right)^z}{\left(1+\dfrac{z}{n}\right)} \quad .$$

By Theorem 5.4, we have $\zeta(\aleph_{\mathcal{X}} + b) = 1$ so we may use the functional form of ζ to compute $\zeta(-\aleph_{\mathcal{X}} + b')$ for $b' = 1 - b$. We have

$$\begin{aligned}
\Gamma(\aleph_{\mathcal{X}} + b) &= \frac{1}{\aleph_{\mathcal{X}} + b} \prod_{n=1}^{\widehat{\infty}} \left(1 + \aleph_{(\mathcal{X}/n)} + \frac{b}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{\aleph_{\mathcal{X}}+b} \\
&= (0) \prod_{n=1}^{\widehat{\infty}} (0) \left(1+\frac{1}{n}\right)^{\aleph_{\mathcal{X}}+b} \\
&= \prod_{n=1}^{\widehat{\infty}} \left(\frac{0}{n}\right)^{\aleph_{\mathcal{X}}+b} = 0 \quad .
\end{aligned}$$

It follows that

$$\begin{aligned}
\zeta(-\aleph_{\mathcal{X}} + b') &= 2(2\pi)^{-\aleph_{\mathcal{X}}+b'-1} \sin\left[\frac{\pi(-\aleph_{\mathcal{X}} + b')}{2}\right] \Gamma(\aleph_{\mathcal{X}} + b)\zeta(\aleph_{\mathcal{X}} + b) \\
&= \frac{2(2\pi)^{b'-1}}{[(2\pi)^{\widehat{\infty}}]^{\mathcal{X}}} \sin\left[\frac{\pi(-\aleph_{\mathcal{X}} + b')}{2}\right] (0)(1) = 0 \quad .
\end{aligned}$$

Theorem 5.6 *The Riemann ζ function has non-trivial zeros with negative real parts in $\widehat{\mathbb{R}}$.*

Proof. By letting $\mathcal{X} = 1$ and choosing an appropriate b, proof follows from Theorem 5.5.

Remark 5.7 To demonstrate that Riemann's functional form of ζ is robust, we should check for consistency by reversing the sign of z and $1 - z$ to show that there is no contradiction. We have

$$\Gamma(-\aleph_{\mathcal{X}} + 1) = \frac{1}{-\aleph_{\mathcal{X}} + 1} \prod_{n=1}^{\widehat{\infty}} \left(1 - \aleph_{(\mathcal{X}/n)} + \frac{1}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{-\aleph_{\mathcal{X}}+1} = 0 \quad .$$

Evaluation of $\zeta(\aleph_\mathcal{X})$ yields

$$\zeta(\aleph_\mathcal{X}) = 2(2\pi)^{\aleph_\mathcal{X}-1}\sin\left(\frac{\pi\aleph_\mathcal{X}}{2}\right)\Gamma(-\aleph_\mathcal{X}+1)\zeta(-\aleph_\mathcal{X}+1) = (\widehat{\infty})(0)(0) \quad .$$

This equation is undefined and we cannot obtain a contradiction.

Theorem 5.8 *The Euler product from of ζ has non-trivial zeros with negative real parts in $\mathbb{R}_\infty$.*

Proof. Consider a number $z_0 \in \mathbb{C}$ such that

$$z_0 = -(\aleph_\mathcal{X} - b) + iy_0 \quad , \qquad \text{where} \qquad b, y_0 \in \mathbb{R}_0 \quad .$$

Observe that the Euler product form of ζ [9] takes z_0 as

$$\begin{aligned}
\zeta(z_0) &= \prod_p \frac{1}{1 - p^{(\aleph_\mathcal{X}-b)-iy_0}} \\
&= \left(\frac{1}{1 - P^{(\mathcal{X}\aleph_1 - b)-iy_0}}\right)\prod_{p\neq P}\frac{1}{1 - p^{(\aleph_\mathcal{X}-b)-iy_0}} \\
&= \left(\frac{1}{1 - \dfrac{1}{P^b}(P^{\widehat{\infty}})^\mathcal{X}\left[\cos(y_0\ln P) - i\sin(y_0\ln P)\right]}\right)\prod_{p\neq P}\frac{1}{1 - p^{(\aleph_\mathcal{X}-b)-iy_0}} \quad .
\end{aligned}$$

Let $y_0 \ln P = 2n\pi$ for some prime P and $n \in \mathbb{N}$ or $n = 0$. Then

$$\zeta(z_0) = \left(\frac{1}{1-\widehat{\infty}}\right)\prod_{p\neq P}\frac{1}{1 - p^{(\aleph_\mathcal{X}-b)-iy_0}} = 0 \quad .$$

Remark 5.9 In this section, we have cleanly demonstrated the negation of the Riemann hypothesis. Although ζ does not absolutely converge to the Euler product in the left complex half-plane, the case of $n = 0$ in Theorem 5.8 shows that the product form of ζ is exactly equal to the sum form for some z with $\text{Re}(z) < 0$. Therefore, we have good reason to assume that the complex zeros defined by $n \neq 0$ in Theorem 5.8 are non-trivial zeros of the Riemann ζ function.

§6 The Modified Cosmological Model

Here we come to the main relevance of the present analysis to the physical model which motivated the underlying philosophical inquiry into fractional

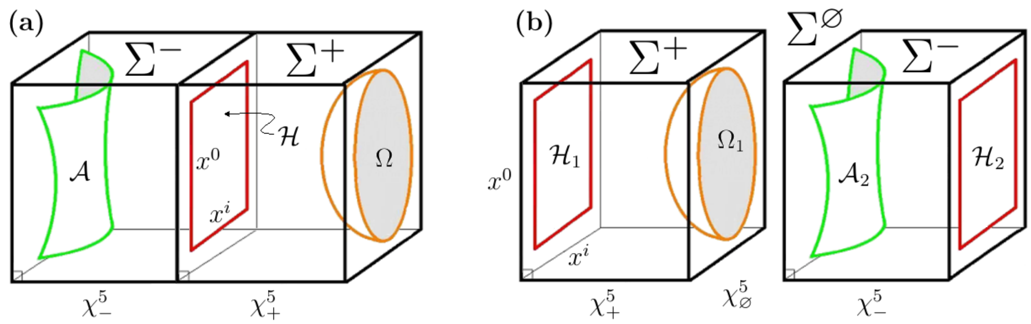

Figure 2: This figure shows the MCM unit cell represented in two different periodicities.

distance. The Modified Cosmological Model is detailed at length in Reference [2]. Two short papers, References [10] and [11], very efficiently motivate the physical utility of Figures 2(a) and 2(b) which show the unit cell of the MCM cosmological lattice. The most recent treatment of the MCM unit cell, one which makes more precisely mathematical the properties detailed in earlier work, appears in Reference [12]. One of the main unresolved issues in the MCM regards Figure 2(b) and the disconnection between $\Sigma^{\pm}$. A primary application of the results of the present paper to the MCM should be to assign a number $\mathcal{F}_n$ as the value of $\chi^5_{\varnothing}$ at the location of $\Sigma^{\varnothing}$.

In 1921, Kaluza showed how gravity and electromagnetism may be unified in the metric of a 5D space which contains only 4D matter-energy [13, 14]. The Kaluza metric, also called the Kaluza–Klein (KK) metric when the fifth dimension is compactified via some exponential map, is

$$g_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi^2 A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \kappa\phi^2 A_\beta & \phi^2 \end{pmatrix} \quad .$$

The unified KK theory does not quite work because it requires that the electromagnetic (EM) field strength tensor

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad ,$$

must vanish at all times [14]. Obviously, this restriction is unphysical. In Reference [10], we proposed to solve this problem via the introduction of two 5D spaces $\Sigma^{\pm}$ such that two 5D metrics contain four different electromagnetic potential vectors $A^{\pm}_\alpha$ and $A^{\pm}_\beta$. In this way, the KK constraint requiring a vanishing EM strength tensor can be alleviated by doubling the degrees of freedom contained in the EM potential vectors. While we have not yet shown precisely the totally unified field dynamics, we have proven by principle of sufficient reason that that which is impossible with two EM potential vectors should be possible with the added freedom contained in four EM potential vectors... should anyone seek to undertake the putting together of such a non-trivial set of field equations.

To build the MCM unit cell depicted in Figure 2, we require that $\Sigma^{\pm}$ are strictly 5D half-spaces: Σ^{+} contains only positive values of the fifth coordinate χ^5, and Σ^{-} only negative. We impose a boundary condition such that $\mathcal{H}$ is a 4D Minkowski space which sews together $\Sigma^{\pm}$ where the fifth coordinate is equal to 0. Since the EM potential vector affects the 4-metric in the $\alpha\beta$ part of g_{AB}, we must require that $A^{\pm}_{\alpha}$ and $A^{\pm}_{\beta}$ are equal to 0 when $\mathcal{H}$ is an identically flat Minkowski space. This defines the ground state of the MCM only. It is not a global constraint such as the vanishing EM tensor in KK theory. If we allow non-vanishing EM potential vectors then this will induce perturbations on the metric of $\mathcal{H}$ which will, in turn, disrupt the simplifying symmetries of the unit cell.

By adding the second 5-space, we have shown in the MCM how to solve the problem of classical electrogravity [10]. Another problem in modern physics is how to couple quantum field theory to the geometry of spacetime. Since elementary QFT is merely quantized electromagnetism, we should find this coupling via the KK mechanism which unifies EM and gravitation. Specifically, to show that a QFT is coupled to the geometry of spacetime, we need to apply some evolution operator to an initial state existing in a given metrical background, and then obtain a simultaneous final state and final metric at some later time. We set the stage for this in the unit cell with an initial condition in $\mathcal{H}_1$ and a final condition in $\mathcal{H}_2$. The first step toward solving the coupling problem is to evolve a vacuum state in flat spacetime from one moment ($\mathcal{H}_1$) to a later moment ($\mathcal{H}_2$) where the vacuum continues to exist in flat spacetime. After propagation of the vacuum is demonstrated, one would define a non-vacuum state and show how the evolution of the state perturbs the local spacetime metric from one instance of $\mathcal{H}$ to the next. To that end, we define the operator

$$\hat{M}^3 : \mathcal{H}_1 \to \Omega_1 \to \mathcal{A}_2 \to \mathcal{H}_2 \quad .$$

An outstanding question is how exactly to evolve from Ω_1 to $\mathcal{A}_2$. To see the difficulty of the issue we will examine the metrics of $\Sigma^{\pm}$ when $A_\alpha = A_\beta = 0$, and also specify an interpretation for the scalar field ϕ^2. The metrics are

$$\Sigma^{\pm}_{AB} = \begin{pmatrix} g^{\pm}_{\alpha\beta} & 0 \\ 0 & \phi^2(\chi^5_{\pm}) \end{pmatrix} \quad ,$$

where $\chi^A_{\pm}$ are the 5D coordinates in $\Sigma^{\pm}$. It is a condition inherited from KK theory that $\phi^2_{\pm}$ can only be a function of $\chi^5_{\pm}$. If $\phi^2_{\pm}$ depends on any of $\chi^\alpha_{\pm}$ then electromagnetism and gravitation are not unified. For the MCM, we take ϕ^2 as the de Sitter parameter K of curvature such that each slice of constant χ^5 is a maximally symmetric spacetime of constant curvature given by a function of χ^5 only. To make this perfectly clear, we may express $\Sigma^\pm_{AB}$ as the canonical de Sitter metric of maximally symmetric 4D spacetime:

$$\Sigma^\pm_{AB} = \begin{pmatrix} g^\pm_{\alpha\beta} & 0 \\ 0 & K_\pm(\chi^5_\pm) \end{pmatrix} \quad .$$

Minkowski space has zero curvature. Since $\mathcal{H}$ is located at $\chi^5_\pm = 0$, this suggests that K should be some scalar multiple of χ^5. Because Σ^+ is only a 5D half-space, χ^5_+ is positive everywhere in Σ^+ and all its slices of constant χ^5_+ are 4D de Sitter spaces. Likewise in Σ^-, χ^5_- is everywhere negative and the slices of constant χ^5_- are 4D anti-de Sitter spaces.

Now that we have clearly defined the objects in Figure 2, we can see how Σ^- can be smoothly sewn to Σ^+ in the direction of increasing $\chi^5_\pm$. The magnitude of the curvature of the slices decreases in a monotone way from $\mathcal{A}$ down to 0 at $\mathcal{H}$, and then it increases smoothly to Ω across Σ^+. However, to connect $\Sigma^\pm$ as in Figure 2(b), it is not immediately clear how we might smoothly connect a space with large positive curvature to a space with large negative curvature. The idea that we have come up with [15] is to let the curvature tend to $\pm\infty$ on Ω and $\mathcal{A}$ respectively so that the curvature tends to an infinite value as the 5-spaces close down to a singularity in $\Sigma^\varnothing$. At that point, we can model the forward evolution from Σ^+ to Σ^- as falling into a black hole and then being ejected on the other side from a white hole. Such mechanisms are well defined.

Without getting too much into the physics, especially considering that the model is still in development and is not yet at any level of rigor even approaching that used in the previous sections, the utility of fractional distance toward the MCM is this: When the unit cell is centered on $\mathcal{H}$, the unit cell's spanning interval of $\chi^5_\pm$ should be like some $\mathbb{R}^{\mathcal{X}}_{\aleph}$ and when the unit cell is centered on $\Sigma^\varnothing$ the spanning interval of $\chi^5_\mp$ should be like some $\mathbb{R}^n_{\mathcal{F}}$. Indeed, where we have shown in Section 3.4 that numbers such as K reverse their behavior under the Archimedes property between the neighborhood of the origin in $\mathcal{H}$ and the neighborhood of infinity at other side of $\Sigma^\pm$, that should directly motivate the time reversal needed to extract an outgoing white hole solution in Σ^- from and infalling black hole solution in Σ^+. Certainly, there are very many details to be clarified but the purpose of this closing section is only to mention some likely avenues for forward progress with the outstanding issues of the MCM.

Regarding $\hat{M}^3$, we have not yet defined the enumeration scheme for defining adjacent $\mathbb{R}^{\mathcal{X}}_{\aleph}$ neighborhoods but we would need to do so if we were going to place $\mathcal{H}_1$ and $\mathcal{H}_2$ in two such adjacent neighborhoods separated by $\Sigma^\varnothing$ at some $\mathcal{F}_n$. Rather than dealing with $\mathcal{F}_n$, it might be simpler, however, to define some transfinite scheme of analytic continuation such that the evolution operator $\hat{M}^3$ is defined as

$$\hat{M}^3 : \mathbb{R}_0 \to \widehat{\mathbb{R}}^+ \to \widehat{\mathbb{R}}^- \to \mathbb{R}_0 \quad .$$

In this case, we would place algebraic infinity $\widehat{\infty}$ at the location of $\Sigma^\varnothing$ and then simply do a continuation onto the extent of $\mathbb{R}$ that exists between $\widehat{\infty}$ in some local chart and ∞ in the absolute parent chart.

Another likely utility for the tools developed in this paper regards the compactification mechanism which distinguishes KK theory from non-compactified Kaluza theory. If we choose a metric of the form

$$\Sigma^{\pm}_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \aleph_{(c_\pm \chi^5_\pm)} \end{pmatrix} \quad ,$$

then the limit of the curvature as $\chi^5 \to 0$ will be zero, as required, on approach to $\mathcal{H}$. Since $\chi^5_\pm \neq 0$ in $\Sigma^\pm$, this definition avoids the undefined number $\aleph_0$. If we choose $c_\pm$ such that Ω is located at $\chi^5_+ = c_+^{-1}$ and $\mathcal{A}$ is located at $\chi^5_- = c_-^{-1}$, then Ω and $\mathcal{A}$ will become topological singularities of infinite positive and negative curvature respectively. By using an $\aleph_{\mathcal{X}}$ number in the fifth metric position, we have "megafied" that dimension rather than compactifying it, but the principle of compactification is preserved: 4D universes do not interact strongly with the fifth dimension because they are of radically different scale.

Another likely avenue for strongly productive forward development regards the SU(2) part of the electroweak theory. It is said that the degrees of freedom of this theory behave as if there is a small 2-sphere embedded at every point of spacetime. This sphere has two degrees of freedom on its surface which are accessible to quantum particles. Since the quantum electroweak theory uses complex numbers, we are motivated to invoke distinct real and complex infinities. If we replace the notion of "every point in spacetime" with the notion of "every point in the neighborhood of the origin," then we can construct two additional degrees of freedom by sewing together the disjoint subsets of $\widehat{\mathbb{R}}$ and $i\widehat{\mathbb{R}}$. Since every point within an $\mathbb{R}_0$ radius of the origin of $\mathbb{C}$ is surrounded by the neighborhood of infinity, the ubiquity of these two degrees of freedom across all points in spacetime is the same as the ubiquity generated by embedding a tiny 2-sphere at every point. Indeed, if we define two intervals of freedom

$$D_1 = \widehat{\mathbb{R}}^+ \cup \widehat{\mathbb{R}}^- \quad , \qquad \text{and} \qquad D_2 = i\widehat{\mathbb{R}}^+ \cup i\widehat{\mathbb{R}}^- \quad ,$$

then we may use the exponential map to send D_1 and D_2 to two great circles of $\mathbb{S}^2$ which are *exactly* like the SU(2) degrees of freedom in the electroweak theory. We have shown in Section 3.4 that degrees of freedom such as D_1 and D_2 should have reversed behavior under the Archimedes property so we would expect this freedom to display the opposite variational phase which is characteristic of the fermionic particles that electroweak theory was invented to describe.

The final thing we will mention here is the concept of levels of aleph. In the MCM, we introduce levels of aleph as an analogue of hyperreal numbers [16, 17]. In the present work, when defining the $\mathbb{R}^{\mathcal{X}}_{\aleph}$ neighborhoods, we gave the same linear scale to the b in any number of the form $x = \aleph_{\mathcal{X}} + b$. To adapt this concept to what we have called in the MCM "levels of aleph," the units of b, or the linear scale of those units rather, would need to grow with increasing $\mathcal{X}$. Since we have shown that each $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is pointlike in **AB** and has zero fractional distance, we are free to increase the linear scale by any factor because the fractional distance will remain zero. If we tried to formulate levels of aleph exactly as the hyperreals such that an infinitesimal was like an infinite on a higher tier of infinitude, then we would not be able to achieve this growth with a linear scale factor. However, levels of aleph differ from hyperreals by adding an intermediate scale such that finites are like infinites on the next tier of infinitude, or the next "level of aleph." In this way, we should be able to redefine the $\mathbb{R}_{\aleph}^{\mathcal{X}}$ to accommodate the requisite growth in the units of b from one level of aleph to the next.

To see why the linear scale of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ should grow, consider the polar ray in the construction of the Riemann sphere. If $\theta = 0$ when the polar ray points to the origin of $\mathbb{R}$, then θ will increase as the polar ray approaches the neighborhoods of $\aleph_{\mathcal{X}}$. Although we have some idea that θ will reach $\pi/2$ by the time the sweep of the polar ray would leave the $\mathbb{R}_0$ neighborhood of the origin, it also follows that the linear distance along $\mathbb{R}$ covered by the polar ray across each increasing neighborhood of $\aleph_{\mathcal{X}}$ would be infinitely large compared to the $\mathbb{R}_{\aleph}^{\mathcal{X}}$ closer to the origin. For this reason, we would be well motivated to increase the linear scale of the b in each successive neighborhood. Although an increase of one level of aleph is associated with $1 \to \infty$ rather than the $0 \to \infty$ change of scale associated with the polar ray, we have not mentioned the odd and even levels of aleph. When the polar ray sweeps through $\mathbb{R}_{\aleph}^{\mathcal{X}}$, then $\mathbb{R}_{\mathcal{F}}^{n}$, then another $\mathbb{R}_{\aleph}^{\mathcal{X}}$, that will increase the level of aleph by two rather than by one, and the attendant change of scale would be $0 \to 1 \to \infty$, exactly as expected. Once we generate an increasing scale for the b in each $\mathbb{R}_{\aleph}^{\mathcal{X}}$, we may define an extension of the natural numbers so that they do not end in $\mathbb{R}_{\aleph}^{0}$ but rather keep going forever. Specifically, the number 1 defined by the scale of some $\mathbb{R}_{\aleph}^{\mathcal{X}_2}$ would be greater than the largest natural number defined by the scale of $\mathbb{R}_{\aleph}^{\mathcal{X}_1}$ whenever $\mathcal{X}_2 > \mathcal{X}_1$. This extension of the naturals would restore the property of natural numbers such that **there is no real number greater than every extended natural number**. With such an extended set of natural numbers in place, we would have everything needed to construct the generalized exponential function proposed in Reference [18].

Levels of aleph are such that if $z = x\hat{\Phi}^k$, then z is on the k^{th} level of aleph. To go forward, we must enumerate sequential levels of aleph such that if k is even then $z \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ and if k is odd then $z \in \mathbb{R}_{\mathcal{F}}^{n}$. Without being too specific, we would introduce some notation like $\mathbb{R}_{\Phi}^{k}$ such that

$$z = x\,\hat{\Phi}^k \quad \implies \quad z \in \mathbb{R}_{\Phi}^{k} \quad .$$

Since every $\mathbb{R}_0$ number has the property that $z = z\hat{1}$, and $\hat{1} = \hat{\Phi}^0$, we are well motivated to define the exponential function as

$$e^x = \hat{\mathcal{P}}_0 E^x \quad ,$$

where E^x is the generalized exponential function that sums over the extended natural numbers rather than only $n \in \mathbb{N}$, and $\hat{\mathcal{P}}_k$ is a projection operator selecting the $\hat{\Phi}^k$ component. Specifically,

$$e^x = \sum_{n\in\mathbb{N}} \frac{x^{n-1}}{(n-1)!} \quad , \qquad \text{and} \qquad E^x = \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}_k} \frac{x^{n-1}}{(n-1)!}\hat{\Phi}^{2k-2} \quad .$$

Here $\mathbb{N}_1 = \mathbb{N}$ and other $\mathbb{N}_k$ are the extended natural numbers measured in the scaled units of $\mathbb{R}_\Phi^{(2k-2)} = \mathbb{R}_\aleph^{\mathcal{X}}$ for some $\mathcal{X}$. Since we restrict to even levels of aleph with the $\hat{\Phi}^{(2k-2)}$ notation, we guarantee that the largest number in one sum is smaller than the unit number in the next sum (the unit number being the smallest natural number in a given $\mathbb{N}_k$.) If we chose sequential levels of aleph for the generalized exponential function, then there could be some overlap in the scale when we take the limiting behavior $1 \to \infty$. By choosing only even levels, we prohibit any overlap with the change of linear scale $0 \to 1 \to \infty$. When a wavefunction has the form

$$\psi = \psi_0 e^{i\omega t} \quad ,$$

in $\mathcal{H}_1$ then it will have the form

$$\psi = \psi_0 e^{i\omega t}\,\hat{\Phi}^2 \quad ,$$

when it reaches $\mathcal{H}_2$. Obviously such a wavefunction cannot satisfy the probability interpretation of quantum mechanics which requires

$$\int_{-\widehat{\infty}}^{\widehat{\infty}} \psi^* \psi \, dx = 1 \quad .$$

Therefore, the operator $\hat{M}^3$ would necessarily include some term $\hat{\Phi}^{-2}$ that allows us "to go to" the framework of analysis in which $z = x\hat{\Phi}^2$ behaves like a finite number, and *is* a finite number in that framework.

The purpose of the generalized exponential function in quantum theory will be so that, perhaps, we may define the wavefunction in terms of E^x rather than e^x, and then develop a set of operators which operate directly on E^x without requiring that we first use a projection operator to select the $e^x\hat{\Phi}^k$ single sum over the natural numbers in only the $\mathbb{R}_\Phi^k$ neighborhood of $\mathbb{R}$.

References

[1] Jonathan W. Tooker. Real Numbers in the Neighborhood of Infinity. *viXra:1811.0222*, (2018).

[2] Jonathan W. Tooker. The General Relevance of the Modified Cosmological Model. *viXra:1712.0598*, (2017).

[3] David Hilbert. The Foundations of Geometry. (1899).

[4] Richard Dedekind. Essays on the Theory of Numbers. (1872).

[5] Jonathan W. Tooker. On the Riemann Zeta Function. *viXra:1703.0073*, (2017).

[6] Jin Gyu Lee. Zeros of the Riemann Zeta Function. *arXiv:1305:0323*, (2013).

[7] Bernhard Riemann. On the Number of Primes Less than a Given Quantity. *Monatsberichte der Berliner Akademie*, (1859).

[8] Leonhard Euler. Letter to Christian Goldbach. (1729).

[9] Leonhard Euler. Various Observations about Infinite Series. *St. Petersburg Academy*, (1737).

[10] Jonathan W. Tooker. Geometric Cosmology. *viXra:1301.0032*, (2013).

[11] Jonathan W. Tooker. Quantum Structure. *viXra:1302.0037*, (2013).

[12] Jonathan W. Tooker. The Golden Ratio in the Modified Cosmological Model. *viXra:1807.0136*, (2018).

[13] Theodor Kaluza. The Unification Problem in Physics. *Sitzungsber. Preuss. Akad. Wiss. Berlin., 966-972*, (1921).

[14] J. M. Overduin and P. S. Wesson. Kaluza-Klein Gravity. *arXiv:gr-qc/9805018*, (1998).

[15] Jonathan W. Tooker. The General Relevance of the Modified Cosmological Model, Section IV.2. *viXra:1712.0598*, (2017).

[16] Jonathan W. Tooker. The General Relevance of the Modified Cosmological Model, Section I.1. *viXra:1712.0598*, (2017).

[17] Jonathan W. Tooker. The General Relevance of the Modified Cosmological Model, Section I.2. *viXra:1712.0598*, (2017).

[18] Jonathan W. Tooker. Ontological Physics. *viXra:1312.0168*, (2013).