Quick Disproof of the Riemann Hypothesis

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Abstract

In this brief note, we propose a set of operations for the affinely extended real number called infinity. Under the terms of the proposition, we show that the Riemann zeta function has infinitely many non-trivial zeros on the complex plane.

§1 Definitions

Definition 1.1 Infinity is a number, not a real number, defined as

$$\lim_{x\to 0^\pm}\frac{1}{x}=\pm\infty \quad , \qquad \text{and} \qquad \lim_{n\to\infty}\sum_{k=1}^{n}k=\infty \quad .$$

Definition 1.2 The real number line is a 1D space extending infinitely far in both directions. It is represented in interval notation as

$$\mathbb{R}\equiv(-\infty,\infty) \quad .$$

Definition 1.3 A number x is a real number if and only if it is a cut in the real number line:

$$(-\infty,\infty)=(-\infty,x)\cup[x,\infty) \quad .$$

Definition 1.4 The affinely extended real numbers are constructed as $\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$. They are represented in interval notation as

$$\overline{\mathbb{R}}\equiv[-\infty,\infty] \quad .$$

Definition 1.5 An affinely extended real number $x\in\overline{\mathbb{R}}$ is $\pm\infty$ or it is a cut in the affinely extended real number line:

$$[\infty,\infty]=[-\infty,x)\cup[x,\infty] \quad .$$

Theorem 1.6 *If $x\in\overline{\mathbb{R}}$ and $x\neq\pm\infty$, then $x\in\mathbb{R}$.*

Proof. Proof follows from Definition 1.4.

Definition 1.7 Infinity has the properties of additive and multiplicative absorption:

$$x \in \mathbb{R} \quad , \quad x > 0 \qquad \Longrightarrow \qquad \begin{cases} \pm x + \infty = \infty \\ \pm x \times \infty = \pm\infty \end{cases} \qquad .$$

Proposition 1.8 Suppose the additive absorptive property of $\pm\infty$ is taken away when it appears as $\pm\widehat{\infty}$. Further suppose that $\|\widehat{\infty}\| = \|\infty\|$, and that the ordering of $\widehat{\infty}$ is such that

$$n < \widehat{\infty} - b < \widehat{\infty} - a < \infty$$
$$-\infty < -\widehat{\infty} + a < -\widehat{\infty} + b < -n \quad ,$$

for any positive $a, b \in \mathbb{R}$, $a < b < n$, and any natural number $n \in \mathbb{N}$.

Remark 1.9 Because $\|\widehat{\infty}\| = \|\infty\|$, it is possible to express $\overline{\mathbb{R}}$ in interval notation as

$$\overline{\mathbb{R}} = [-\widehat{\infty}, \widehat{\infty}] \quad .$$

Theorem 1.10 *If $x = \pm(\widehat{\infty} - b)$ and $0 < b < n$ for some $n \in \mathbb{N}$, then $x \in \mathbb{R}$.*

Proof. By the ordering given in Proposition 1.8, we have

$$[\infty, \infty] = [-\infty, x) \cup [x, \infty] \quad .$$

It follows from Definition 1.5 that $x \in \overline{\mathbb{R}}$. Since $\widehat{\infty}$ does not have additive absorption and the theorem states that $b > 0$, it follows from the ordering that

$$x \neq \pm\widehat{\infty} \quad , \qquad \text{and} \qquad x \neq \pm\infty \quad .$$

It follows from Theorem 1.6 that $x \in \mathbb{R}$. ∎

Theorem 1.11 *If $0 < b < n$ for some natural number $n \in \mathbb{N}$, then the quotient $n/(\widehat{\infty} - b)$ is identically zero.*

Proof. For proof by contradiction, let z be any positive real number such that

$$\frac{n}{\widehat{\infty} - b} = z \quad .$$

Proposition 1.8 requires $||n|| < ||(\widehat{\infty} - b)||$ so we have $||z|| < 1$. All non-zero real numbers have a multiplicative inverse. We find, therefore, that

$$\frac{n}{z(\widehat{\infty} - b)} = 1 \qquad \Longleftrightarrow \qquad n = z(\widehat{\infty} - b) \quad .$$

The hat on $\widehat{\infty}$ only suppresses additive absorption so

$$n = (\widehat{\infty} - zb) \quad .$$

This delivers a contradiction because Proposition 1.8 requires that n be less that $(\widehat{\infty} - b)$ while $z < 1$ requires $(\widehat{\infty} - b) < (\widehat{\infty} - zb)$. ∎

Definition 1.12 A number is a complex number $z \in \mathbb{C}$ if and only if

$$z = x + iy \quad , \qquad \text{and} \qquad x, y \in \mathbb{R} \quad .$$

§2 Disproof of the Riemann Hypothesis

Theorem 2.1 *If $0 < b < n$ for some $n \in \mathbb{N}$, $z_0 = \widehat{\infty} - b$, and $\zeta(z)$ is the Riemann ζ function, then $\zeta(z_0) = 1$.*

Proof. Observe that the Dirichlet sum form of ζ [1]

$$\zeta(z) = \sum_n \frac{1}{n^z} \quad ,$$

takes z_0 as

$$\zeta(\widehat{\infty} - b) = \sum_{n=1} \frac{1}{n^{\widehat{\infty}-b}} = \sum_{n=1} \frac{n^b}{n^{\widehat{\infty}}} = 1 + \sum_{n=2} \frac{n^b}{\widehat{\infty}} = 1 \quad .$$ ∎

Theorem 2.2 *The Riemann ζ function has non-trivial zeros at certain $z \in \mathbb{C}$ outside of the critical strip.*

Proof. Riemann's functional form of ζ [1] is

$$\zeta(z) = 2(2\pi)^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

We have solved for $\zeta(\widehat{\infty} - b)$ in Theorem 2.1 so we will use Riemann's equation to prove this theorem by solving for $z = -(\widehat{\infty} - b) + 1$. To do so, we will use the Euler definition of the Γ function [2]

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\widehat{\infty}} \frac{\left(1 + \frac{1}{n}\right)^z}{\left(1 + \frac{z}{n}\right)} \quad .$$

By Theorem 2.1, we have $\zeta(\widehat{\infty}-b)=1$ so we may use the functional form of ζ to compute $\zeta(-\widehat{\infty}+b+1)$. Using Theorem 1.11, we have

$$\begin{aligned}\Gamma(\widehat{\infty}-b) &= \frac{1}{\widehat{\infty}-b}\prod_{n=1}^{\widehat{\infty}}\left(1+\widehat{\infty}+\frac{b}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{\widehat{\infty}-b} \\ &= (0)\prod_{n=1}^{\widehat{\infty}}(0)\left(1+\frac{1}{n}\right)^{\widehat{\infty}-b} \\ &= \prod_{n=1}^{\widehat{\infty}}\left(\frac{0}{n}\right)^{\widehat{\infty}-b} = 0 \quad .\end{aligned}$$

It follows that

$$\begin{aligned}\zeta(-\widehat{\infty}+b+1) &= 2(2\pi)^{-\widehat{\infty}+b}\sin\left[\frac{\pi(-\widehat{\infty}+b+1)}{2}\right]\Gamma(\widehat{\infty}-b)\zeta(\widehat{\infty}-b) \\ &= \frac{2(2\pi)^b}{(2\pi)^{\widehat{\infty}}}\sin\left[\frac{\pi(-\widehat{\infty}+b+1)}{2}\right](0)(1) = 0 \quad .\end{aligned}$$

Remark 2.3 Since we have shown that $\zeta(-\widehat{\infty}+b+1)$ is equal to zero for any positive $b\in\mathbb{R}$ less than some natural number, most of the zeros shown in Theorem 2.2 cannot be what are called trivial zeros. Theorem 1.10 proves $z_0\in\mathbb{R}$, and it follows from Definition 1.12 that $z_0\in\mathbb{C}$. Since these zeros do not lie inside the critical strip, Theorem 2.2 is the negation of the Riemann hypothesis.

Remark 2.4 To demonstrate that Riemann's functional form of ζ is robust, and that Proposition 1.8 is sound, we should switch the $\pm\widehat{\infty}$ that appear on the left and right sides of the functional form of ζ

$$\zeta(\widehat{\infty}-b) = 2(2\pi)^{\widehat{\infty}-b-1}\sin\left[\frac{\pi(\widehat{\infty}-b)}{2}\right]\Gamma(-\widehat{\infty}+b+1)\zeta(-\widehat{\infty}+b+1) \quad .$$

The Γ function evaluates to

$$\Gamma(-\widehat{\infty}+b+1) = \frac{1}{-\widehat{\infty}+b+1}\prod_{n=1}^{\widehat{\infty}}\left(1-\widehat{\infty}+\frac{b+1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{-\widehat{\infty}+1} = 0 \quad ,$$

so

$$\zeta(\widehat{\infty}-b) = 1 = 2(2\pi)^{\widehat{\infty}-b-1}\sin\left[\frac{\pi(\widehat{\infty}-b)}{2}\right](0)(0) = (\widehat{\infty})(0) \quad .$$

The right hand side of this equation is undefined so we do not obtain a contradiction.

Theorem 2.5 *The Euler product from of ζ has zeros at certain $z \in \mathbb{C}$ with negative real parts.*

Proof. Consider a number $z_0 \in \mathbb{C}$ such that

$$z_0 = -(\widehat{\infty} - b) + iy_0 \quad , \qquad \text{where} \qquad b, y_0 \in \mathbb{R}_0 \quad .$$

Observe that the Euler product form of ζ [3] takes z_0 as

$$\begin{aligned} \zeta(z_0) &= \prod_p \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \\ &= \left(\frac{1}{1 - P^{(\widehat{\infty}-b)-iy_0}}\right) \prod_{p \neq P} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \\ &= \left(\frac{1}{1 - \frac{1}{P^b} P^{\widehat{\infty}} \left[\cos(y_0 \ln P) - i \sin(y_0 \ln P)\right]}\right) \prod_{p \neq P} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \quad . \end{aligned}$$

Let $y_0 \ln P = 2n\pi$ for some prime P and $n \in \mathbb{N}$ or $n = 0$. Theorem 1.11 gives

$$\zeta(z_0) = \left(\frac{1}{1 - \widehat{\infty}}\right) \prod_{p \neq P} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} = 0 \quad .$$

Remark 2.6 Although ζ does not absolutely converge to the Euler product in the left complex half-plane, the case of $n = 0$ in Theorem 2.5 shows that the product form of ζ is exactly equal to the sum form for some z with $\text{Re}(z) < 0$. Therefore, we have good reason to assume that the complex zeros defined by $n \neq 0$ in Theorem 2.5 are non-trivial zeros of the Riemann ζ function.

References

[1] Bernhard Riemann. On the Number of Primes Less than a Given Quantity. *Monatsberichte der Berliner Akademie*, (1859).

[2] Leonhard Euler. Letter to Christian Goldbach. (1729).

[3] Leonhard Euler. Various Observations about Infinite Series. *St. Petersburg Academy*, (1737).